GiNaC 1.8.10

An open framework for symbolic computation within the C++ programming language
[No value for "UPDATED"]

https://www.ginac.de

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1 Introduction

The motivation behind GiNaC derives from the observation that most present day computer algebra systems (CAS) are linguistically and semantically impoverished. Although they are quite powerful tools for learning math and solving particular problems they lack modern linguistic structures that allow for the creation of large-scale projects. GiNaC is an attempt to overcome this situation by extending a well established and standardized computer language (C++) by some fundamental symbolic capabilities, thus allowing for integrated systems that embed symbolic manipulations together with more established areas of computer science (like computation-intense numeric applications, graphical interfaces, etc.) under one roof.

The particular problem that led to the writing of the GiNaC framework is still a very active field of research, namely the calculation of higher order corrections to elementary particle interactions. There, theoretical physicists are interested in matching present day theories against experiments taking place at particle accelerators. The computations involved are so complex they call for a combined symbolical and numerical approach. This turned out to be quite difficult to accomplish with the present day CAS we have worked with so far and so we tried to fill the gap by writing GiNaC. But of course its applications are in no way restricted to theoretical physics.

This tutorial is intended for the novice user who is new to GiNaC but already has some background in C++ programming. However, since a hand-made documentation like this one is difficult to keep in sync with the development, the actual documentation is inside the sources in the form of comments. That documentation may be parsed by one of the many Javadoc-like documentation systems. If you fail at generating it you may access it from the GiNaC home page (`https://www.ginac.de/reference/`). It is an invaluable resource not only for the advanced user who wishes to extend the system (or chase bugs) but for everybody who wants to comprehend the inner workings of GiNaC. This little tutorial on the other hand only covers the basic things that are unlikely to change in the near future.

1.1 License

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2 A Tour of GiNaC

This quick tour of GiNaC wants to arise your interest in the subsequent chapters by showing off a bit. Please excuse us if it leaves many open questions.

2.1 How to use it from within C++

The GiNaC open framework for symbolic computation within the C++ programming language does not try to define a language of its own as conventional CAS do. Instead, it extends the capabilities of C++ by symbolic manipulations. Here is how to generate and print a simple (and rather pointless) bivariate polynomial with some large coefficients:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

int main()
{
    symbol x("x"), y("y");
    ex poly;

    for (int i=0; i<3; ++i)
        poly += factorial(i+16)*pow(x,i)*pow(y,2-i);

    cout << poly << endl;
    return 0;
}
```

Assuming the file is called `hello.cc`, on our system we can compile and run it like this:

```
$ c++ hello.cc -o hello -lginac -lcln
$ ./hello
355687428096000*x*y+20922789888000*y^2+6402373705728000*x^2
```

(See Appendix B [Package tools], page 116, for tools that help you when creating a software package that uses GiNaC.)

Next, there is a more meaningful C++ program that calls a function which generates Hermite polynomials in a specified free variable.

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

ex HermitePoly(const symbol & x, int n)
{
    ex HKer=exp(-pow(x, 2));
    // uses the identity H_n(x) == (-1)^n exp(x^2) (d/dx)^n exp(-x^2)
    return normal(pow(-1, n) * diff(HKer, x, n) / HKer);
}

int main()
{
    symbol z("z");
```

```
    for (int i=0; i<6; ++i)
        cout << "H_" << i << "(z) == " << HermitePoly(z,i) << endl;

    return 0;
}
```

When run, this will type out

```
H_0(z) == 1
H_1(z) == 2*z
H_2(z) == 4*z^2-2
H_3(z) == -12*z+8*z^3
H_4(z) == -48*z^2+16*z^4+12
H_5(z) == 120*z-160*z^3+32*z^5
```

This method of generating the coefficients is of course far from optimal for production purposes.

In order to show some more examples of what GiNaC can do we will now use the `ginsh`, a simple GiNaC interactive shell that provides a convenient window into GiNaC's capabilities.

2.2 What it can do for you

After invoking `ginsh` one can test and experiment with GiNaC's features much like in other Computer Algebra Systems except that it does not provide programming constructs like loops or conditionals. For a concise description of the `ginsh` syntax we refer to its accompanied man page. Suffice to say that assignments and comparisons in `ginsh` are written as they are in C, i.e. `=` assigns and `==` compares.

It can manipulate arbitrary precision integers in a very fast way. Rational numbers are automatically converted to fractions of coprime integers:

```
> x=3^150;
369988485035126972924700782451696644186473100389722973815184405301748249
> y=3^149;
123329495011708990974900260817232214728824366796574324605061468433916083
> x/y;
3
> y/x;
1/3
```

Exact numbers are always retained as exact numbers and only evaluated as floating point numbers if requested. For instance, with numeric radicals is dealt pretty much as with symbols. Products of sums of them can be expanded:

```
> expand((1+a^(1/5)-a^(2/5))^3);
1+3*a+3*a^(1/5)-5*a^(3/5)-a^(6/5)
> expand((1+3^(1/5)-3^(2/5))^3);
10-5*3^(3/5)
> evalf((1+3^(1/5)-3^(2/5))^3);
0.33408977534118624228
```

The function `evalf` that was used above converts any number in GiNaC's expressions into floating point numbers. This can be done to arbitrary predefined accuracy:

```
> evalf(1/7);
0.14285714285714285714
> Digits=150;
150
> evalf(1/7);
0.142857142857142857
```

```
571428571428571428
```

```
> expand(a*b);
8*x^5*y+17*x^4*y^2+43*x^2*y^4-24*x*y^5+16*x^3*y^3+3*y^6+x^6
> factor(%);
(4*x*y+x^2-y^2)^2*(x^2+3*y^2)
> collect(a+b,x);
4*x^3*y-y^2-3*y^4+(12*y^3+4*y)*x+x^4+x^2*(1+2*y^2)
> collect(a+b,y);
12*x*y^3-3*y^4+(-1+2*x^2)*y^2+(4*x+4*x^3)*y+x^2+x^4
> normal(a/b);
3*y^2+x^2
```

Here we have made use of the `ginsh`-command `%` to pop the previously evaluated element from `ginsh`'s internal stack.

You can differentiate functions and expand them as Taylor or Laurent series in a very natural syntax (the second argument of `series` is a relation defining the evaluation point, the third specifies the order):

```
> diff(tan(x),x);
tan(x)^2+1
> series(sin(x),x==0,4);
x-1/6*x^3+Order(x^4)
> series(1/tan(x),x==0,4);
x^(-1)-1/3*x-1/45*x^3+Order(x^4)
> series(tgamma(x),x==0,3);
x^(-1)-Euler+(1/12*Pi^2+1/2*Euler^2)*x+
(-1/3*zeta(3)-1/12*Pi^2*Euler-1/6*Euler^3)*x^2+Order(x^3)
> evalf(%);
x^(-1)-0.5772156649015328606+(0.9890559953279725555)*x
-(0.9074790760808862905)*x^2+Order(x^3)
> series(tgamma(2*sin(x)-2),x==Pi/2,6);
-(x-1/2*Pi)^(-2)+(-1/12*Pi^2-1/2*Euler^2-1/240)*(x-1/2*Pi)^2
-Euler-1/12+Order((x-1/2*Pi)^3)
```

Often, functions don't have roots in closed form. Nevertheless, it's quite easy to compute a solution numerically, to arbitrary precision:

```
> Digits=50:
> fsolve(cos(x)==x,x,0,2);
0.73908513321516064165531208767387340401341175890075746496 58
> f=exp(sin(x))-x:
> X=fsolve(f,x,-10,10);
2.219107148913746032595785188204290168175366556532067885415 5
> subs(f,x==X);
-6.372367644529809108115521591070847222364418220770475144296E-58
```

Notice how the final result above differs slightly from zero by about $6*10^(-58)$. This is because with 50 decimal digits precision the root cannot be represented more accurately than `X`. Such inaccuracies are to be expected when computing with finite floating point values.

If you ever wanted to convert units in C or C++ and found this is cumbersome, here is the solution. Symbolic types can always be used as tags for different types of objects. Converting from wrong units to the metric system is now easy:

```
> in=.0254*m;
0.0254*m
> lb=.45359237*kg;
0.45359237*kg
```

```
> 200*lb/in^2;
140613.91592783185568*kg*m^(-2)
```

3 Installation

GiNaC's installation follows the spirit of most GNU software. It is easily installed on your system by three steps: configuration, build, installation.

3.1 Prerequisites

In order to install GiNaC on your system, some prerequisites need to be met. First of all, you need to have a C++-compiler adhering to the ISO standard *ISO/IEC 14882:2014(E)*. We used GCC for development so if you have a different compiler you are on your own. For the configuration to succeed you need a Posix compliant shell installed in `/bin/sh`, GNU `bash` is fine. The `pkgconf` utility is required for the configuration, it can be downloaded from `http://pkgconf.org/`. Last but not least, the CLN library is used extensively and needs to be installed on your system. Please get it from `https://www.ginac.de/CLN/` (it is licensed under the GPL) and install it prior to trying to install GiNaC. The configure script checks if it can find it and if it cannot, it will refuse to continue.

3.2 Configuration

To configure GiNaC means to prepare the source distribution for building. It is done via a shell script called `configure` that is shipped with the sources and was originally generated by GNU Autoconf. Since a configure script generated by GNU Autoconf never prompts, all customization must be done either via command line parameters or environment variables. It accepts a list of parameters, the complete set of which can be listed by calling it with the `--help` option. The most important ones will be shortly described in what follows:

- `--disable-shared`: When given, this option switches off the build of a shared library, i.e. a `.so` file. This may be convenient when developing because it considerably speeds up compilation.
- `--prefix=`*`PREFIX`*: The directory where the compiled library and headers are installed. It defaults to `/usr/local` which means that the library is installed in the directory `/usr/local/lib`, the header files in `/usr/local/include/ginac` and the documentation (like this one) into `/usr/local/share/doc/GiNaC`.
- `--libdir=`*`LIBDIR`*: Use this option in case you want to have the library installed in some other directory than *`PREFIX`*`/lib/`.
- `--includedir=`*`INCLUDEDIR`*: Use this option in case you want to have the header files installed in some other directory than *`PREFIX`*`/include/ginac/`. For instance, if you specify `--includedir=/usr/include` you will end up with the header files sitting in the directory `/usr/include/ginac/`. Note that the subdirectory `ginac` is enforced by this process in order to keep the header files separated from others. This avoids some clashes and allows for an easier deinstallation of GiNaC. This ought to be considered A Good Thing (tm).
- `--datadir=`*`DATADIR`*: This option may be given in case you want to have the documentation installed in some other directory than *`PREFIX`*`/share/doc/GiNaC/`.

In addition, you may specify some environment variables. `CXX` holds the path and the name of the C++ compiler in case you want to override the default in your path. (The `configure` script searches your path for `g++`, `c++`, `aCC`, `cxx`, `cc++`, `clang++`, in that order.) It may be very useful to define some compiler flags with the `CXXFLAGS` environment variable, like optimization, debugging information and warning levels. If omitted, it defaults to `-g -O2`.[1]

[1] The `configure` script is itself generated from the file `configure.ac`. It is only distributed in packaged releases of GiNaC. If you got the naked sources, e.g. from git, you must generate `configure` along with the various `Makefile.in` by using the `autoreconf` utility. This will require a fair amount of support from your local toolchain, though.

The whole process is illustrated in the following two examples. (Substitute `setenv` *`VARIABLE value`* for `export` *`VARIABLE`*`=`*`value`* if the Berkeley C shell is your login shell.)

Here is a simple configuration for a site-wide GiNaC library assuming everything is in default paths:

```
$ export CXXFLAGS="-Wall -O2"
$ ./configure
```

And here is a configuration for a private static GiNaC library with several components sitting in custom places (site-wide GCC and private CLN). The compiler is persuaded to be picky and full assertions and debugging information are switched on:

```
$ export CXX=/usr/local/gnu/bin/c++
$ export CPPFLAGS="$(CPPFLAGS) -I$(HOME)/include"
$ export CXXFLAGS="$(CXXFLAGS) -DDO_GINAC_ASSERT -ggdb -Wall -pedantic"
$ export LDFLAGS="$(LDFLAGS) -L$(HOME)/lib"
$ ./configure --disable-shared --prefix=$(HOME)
```

3.3 Building GiNaC

After proper configuration you should just build the whole library by typing

```
$ make
```

at the command prompt and go for a cup of coffee. The exact time it takes to compile GiNaC depends not only on the speed of your machines but also on other parameters, for instance what value for `CXXFLAGS` you entered. Optimization may be very time-consuming.

Just to make sure GiNaC works properly you may run a collection of regression tests by typing

```
$ make check
```

This will compile some sample programs, run them and check the output for correctness. The regression tests fall in three categories. First, the so called *exams* are performed, simple tests where some predefined input is evaluated (like a pupils' exam). Second, the *checks* test the coherence of results among each other with possible random input. Third, some *timings* are performed, which benchmark some predefined problems with different sizes and display the CPU time used in seconds. Each individual test should return a message '`passed`'. This is mostly intended to be a QA-check if something was broken during development, not a sanity check of your system. Some of the tests in sections *checks* and *timings* may require insane amounts of memory and CPU time. Feel free to kill them if your machine catches fire. Another quite important intent is to allow people to fiddle around with optimization.

By default, the only documentation that will be built is this tutorial in `.info` format. To build the GiNaC tutorial and reference manual in HTML, DVI, PostScript, or PDF formats, use one of

```
$ make html
$ make dvi
$ make ps
$ make pdf
```

Generally, the top-level Makefile runs recursively to the subdirectories. It is therefore safe to go into any subdirectory (`doc/`, `ginsh/`, ...) and simply type `make` *target* there in case something went wrong.

3.4 Installing GiNaC

To install GiNaC on your system, simply type

```
$ make install
```

As described in the section about configuration the files will be installed in the following directories (the directories will be created if they don't already exist):

- `libginac.a` will go into *PREFIX*`/lib/` (or *LIBDIR* if specified) which defaults to `/usr/local/lib/`. So will `libginac.so` unless the configure script was given the option `--disable-shared`. The proper symlinks will be established as well.
- All the header files will be installed into *PREFIX*`/include/ginac/` (or *INCLUDEDIR*`/ginac/`, if specified).
- All documentation (info) will be stuffed into *PREFIX*`/share/doc/GiNaC/` (or *DATADIR*`/doc/GiNaC/`, if *DATADIR* was specified).

For the sake of completeness we will list some other useful make targets: `make clean` deletes all files generated by `make`, i.e. all the object files. In addition `make distclean` removes all files generated by the configuration and `make maintainer-clean` goes one step further and deletes files that may require special tools to rebuild (like the `libtool` for instance). Finally `make uninstall` removes the installed library, header files and documentation[2].

[2] Uninstallation does not work after you have called `make distclean` since the `Makefile` is itself generated by the configuration from `Makefile.in` and hence deleted by `make distclean`. There are two obvious ways out of this dilemma. First, you can run the configuration again with the same *PREFIX* thus creating a `Makefile` with a working '`uninstall`' target. Second, you can do it by hand since you now know where all the files went during installation.

4 Basic concepts

This chapter will describe the different fundamental objects that can be handled by GiNaC. But before doing so, it is worthwhile introducing you to the more commonly used class of expressions, representing a flexible meta-class for storing all mathematical objects.

4.1 Expressions

The most common class of objects a user deals with is the expression `ex`, representing a mathematical object like a variable, number, function, sum, product, etc... Expressions may be put together to form new expressions, passed as arguments to functions, and so on. Here is a little collection of valid expressions:

```
ex MyEx1 = 5;                          // simple number
ex MyEx2 = x + 2*y;                    // polynomial in x and y
ex MyEx3 = (x + 1)/(x - 1);            // rational expression
ex MyEx4 = sin(x + 2*y) + 3*z + 41; // containing a function
ex MyEx5 = MyEx4 + 1;                  // similar to above
```

Expressions are handles to other more fundamental objects, that often contain other expressions thus creating a tree of expressions (See Appendix A [Internal structures], page 113, for particular examples). Most methods on `ex` therefore run top-down through such an expression tree. For example, the method `has()` scans recursively for occurrences of something inside an expression. Thus, if you have declared `MyEx4` as in the example above `MyEx4.has(y)` will find `y` inside the argument of `sin` and hence return `true`.

The next sections will outline the general picture of GiNaC's class hierarchy and describe the classes of objects that are handled by `ex`.

4.1.1 Note: Expressions and STL containers

GiNaC expressions (`ex` objects) have value semantics (they can be assigned, reassigned and copied like integral types) but the operator `<` doesn't provide a well-defined ordering on them. In STL-speak, expressions are '`Assignable`' but not '`LessThanComparable`'.

This implies that in order to use expressions in sorted containers such as `std::map<>` and `std::set<>` you have to supply a suitable comparison predicate. GiNaC provides such a predicate, called `ex_is_less`. For example, a set of expressions should be defined as `std::set<ex, ex_is_less>`.

Unsorted containers such as `std::vector<>` and `std::list<>` don't pose a problem. A `std::vector<ex>` works as expected.

See Section 5.1 [Information about expressions], page 48, for more about comparing and ordering expressions.

4.2 Automatic evaluation and canonicalization of expressions

GiNaC performs some automatic transformations on expressions, to simplify them and put them into a canonical form. Some examples:

```
ex MyEx1 = 2*x - 1 + x;  // 3*x-1
ex MyEx2 = x - x;        // 0
ex MyEx3 = cos(2*Pi);    // 1
ex MyEx4 = x*y/x;        // y
```

This behavior is usually referred to as *automatic* or *anonymous evaluation*. GiNaC only performs transformations that are

- at most of complexity $O(n \log n)$

- algebraically correct, possibly except for a set of measure zero (e.g. x/x is transformed to 1 although this is incorrect for $x = 0$)

There are two types of automatic transformations in GiNaC that may not behave in an entirely obvious way at first glance:

- The terms of sums and products (and some other things like the arguments of symmetric functions, the indices of symmetric tensors etc.) are re-ordered into a canonical form that is deterministic, but not lexicographical or in any other way easy to guess (it almost always depends on the number and order of the symbols you define). However, constructing the same expression twice, either implicitly or explicitly, will always result in the same canonical form.
- Expressions of the form 'number times sum' are automatically expanded (this has to do with GiNaC's internal representation of sums and products). For example

```
ex MyEx5 = 2*(x + y);   // 2*x+2*y
ex MyEx6 = z*(x + y);   // z*(x+y)
```

The general rule is that when you construct expressions, GiNaC automatically creates them in canonical form, which might differ from the form you typed in your program. This may create some awkward looking output ('`-y+x`' instead of '`x-y`') but allows for more efficient operation and usually yields some immediate simplifications.

Internally, the anonymous evaluator in GiNaC is implemented by the methods

```
ex ex::eval() const;
ex basic::eval() const;
```

but unless you are extending GiNaC with your own classes or functions, there should never be any reason to call them explicitly. All GiNaC methods that transform expressions, like `subs()` or `normal()`, automatically re-evaluate their results.

4.3 Error handling

GiNaC reports run-time errors by throwing C++ exceptions. All exceptions generated by GiNaC are subclassed from the standard `exception` class defined in the `<stdexcept>` header. In addition to the predefined `logic_error`, `domain_error`, `out_of_range`, `invalid_argument`, `runtime_error`, `range_error` and `overflow_error` types, GiNaC also defines a `pole_error` exception that gets thrown when trying to evaluate a mathematical function at a singularity.

The `pole_error` class has a member function

```
int pole_error::degree() const;
```

that returns the order of the singularity (or 0 when the pole is logarithmic or the order is undefined).

When using GiNaC it is useful to arrange for exceptions to be caught in the main program even if you don't want to do any special error handling. Otherwise whenever an error occurs in GiNaC, it will be delegated to the default exception handler of your C++ compiler's run-time system which usually only aborts the program without giving any information what went wrong.

Here is an example for a `main()` function that catches and prints exceptions generated by GiNaC:

```
#include <iostream>
#include <stdexcept>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

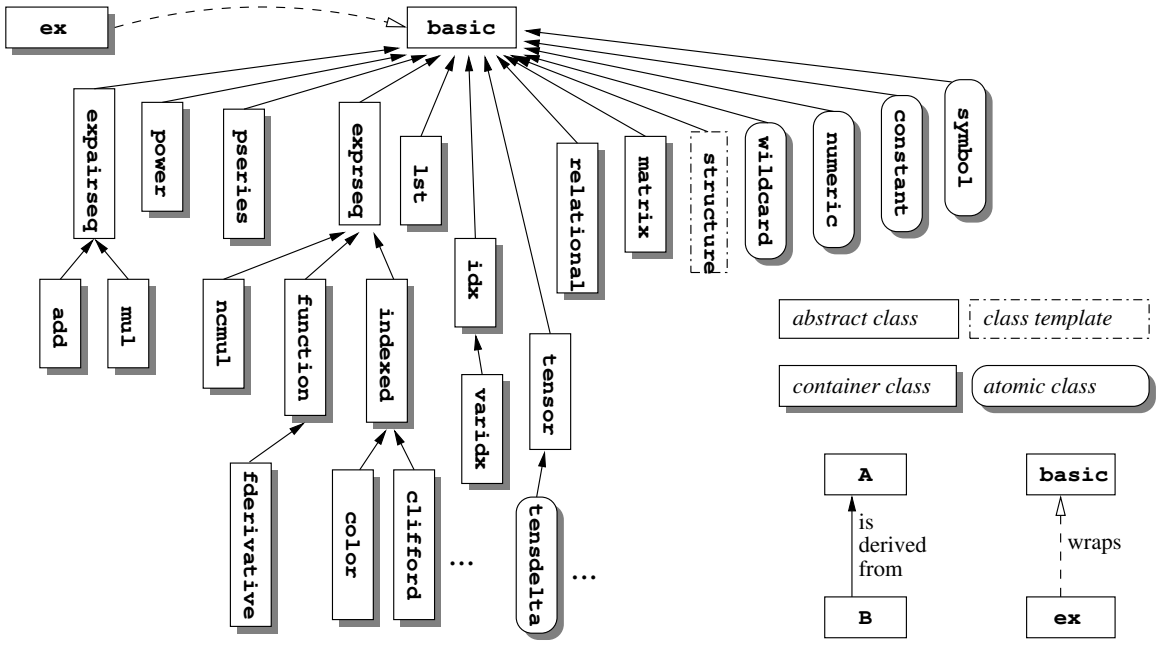
int main()
{
```

```
    try {
        ...
        // code using GiNaC
        ...
    } catch (exception &p) {
        cerr << p.what() << endl;
        return 1;
    }
    return 0;
}
```

4.4 The class hierarchy

GiNaC's class hierarchy consists of several classes representing mathematical objects, all of which (except for `ex` and some helpers) are internally derived from one abstract base class called `basic`. You do not have to deal with objects of class `basic`, instead you'll be dealing with symbols, numbers, containers of expressions and so on.

To get an idea about what kinds of symbolic composites may be built we have a look at the most important classes in the class hierarchy and some of the relations among the classes:



The abstract classes shown here (the ones without drop-shadow) are of no interest for the user. They are used internally in order to avoid code duplication if two or more classes derived from them share certain features. An example is `expairseq`, a container for a sequence of pairs each consisting of one expression and a number (`numeric`). What *is* visible to the user are the derived classes `add` and `mul`, representing sums and products. See Appendix A [Internal structures], page 113, where these two classes are described in more detail. The following table shortly summarizes what kinds of mathematical objects are stored in the different classes:

`symbol`	Algebraic symbols a, x, $y\ldots$
`constant`	Constants like π
`numeric`	All kinds of numbers, 42, $7/3*I$, 3.14159...
`add`	Sums like $x+y$ or $a-(2*b)+3$
`mul`	Products like $x*y$ or $2*a^2*(x+y+z)/b$
`ncmul`	Products of non-commutative objects
`power`	Exponentials such as x^2, a^b, $\sqrt{2}$ ...
`pseries`	Power Series, e.g. $x-1/6*x^3+1/120*x^5+O(x^7)$
`function`	A symbolic function like $\sin 2x$
`lst`	Lists of expressions $\{x, 2*y, 3+z\}$
`matrix`	mxn matrices of expressions
`relational`	A relation like the identity x==y
`indexed`	Indexed object like A_ij
`tensor`	Special tensor like the delta and metric tensors
`idx`	Index of an indexed object
`varidx`	Index with variance
`spinidx`	Index with variance and dot (used in Weyl-van-der-Waerden spinor formalism)
`wildcard`	Wildcard for pattern matching
`structure`	Template for user-defined classes

4.5 Symbols

Symbolic indeterminates, or *symbols* for short, are for symbolic manipulation what atoms are for chemistry.

A typical symbol definition looks like this:

```
symbol x("x");
```

This definition actually contains three very different things:

- a C++ variable named `x`
- a `symbol` object stored in this C++ variable; this object represents the symbol in a GiNaC expression
- the string `"x"` which is the name of the symbol, used (almost) exclusively for printing expressions holding the symbol

Symbols have an explicit name, supplied as a string during construction, because in C++, variable names can't be used as values, and the C++ compiler throws them away during compilation.

It is possible to omit the symbol name in the definition:

```
symbol x;
```

In this case, GiNaC will assign the symbol an internal, unique name of the form `symbolNNN`. This won't affect the usability of the symbol but the output of your calculations will become more readable if you give your symbols sensible names (for intermediate expressions that are only used internally such anonymous symbols can be quite useful, however).

Now, here is one important property of GiNaC that differentiates it from other computer algebra programs you may have used: GiNaC does *not* use the names of symbols to tell them apart, but a (hidden) serial number that is unique for each newly created `symbol` object. If you want to use one and the same symbol in different places in your program, you must only create one `symbol` object and pass that around. If you create another symbol, even if it has the same name, GiNaC will treat it as a different indeterminate.

Observe:

```
ex f(int n)
{
    symbol x("x");
    return pow(x, n);
}

int main()
{
    symbol x("x");
    ex e = f(6);

    cout << e << endl;
     // prints "x^6" which looks right, but...

    cout << e.degree(x) << endl;
     // ...this doesn't work. The symbol "x" here is different from the one
     // in f() and in the expression returned by f(). Consequently, it
     // prints "0".
}
```

One possibility to ensure that `f()` and `main()` use the same symbol is to pass the symbol as an argument to `f()`:

```
ex f(int n, const ex & x)
{
    return pow(x, n);
}

int main()
{
    symbol x("x");

    // Now, f() uses the same symbol.
    ex e = f(6, x);

    cout << e.degree(x) << endl;
     // prints "6", as expected
}
```

Another possibility would be to define a global symbol `x` that is used by both `f()` and `main()`. If you are using global symbols and multiple compilation units you must take special care, however. Suppose that you have a header file `globals.h` in your program that defines a `symbol x("x");`. In this case, every unit that includes `globals.h` would also get its own definition of `x` (because header files are just inlined into the source code by the C++ preprocessor), and hence you would again end up with multiple equally-named, but different, symbols. Instead, the `globals.h` header should only contain a *declaration* like `extern symbol x;`, with the definition of `x` moved into a C++ source file such as `globals.cpp`.

A different approach to ensuring that symbols used in different parts of your program are identical is to create them with a *factory* function like this one:

```
const symbol & get_symbol(const string & s)
{
    static map<string, symbol> directory;
```

```
    map<string, symbol>::iterator i = directory.find(s);
    if (i != directory.end())
        return i->second;
    else
        return directory.insert(make_pair(s, symbol(s))).first->second;
}
```

This function returns one newly constructed symbol for each name that is passed in, and it returns the same symbol when called multiple times with the same name. Using this symbol factory, we can rewrite our example like this:

```
ex f(int n)
{
    return pow(get_symbol("x"), n);
}

int main()
{
    ex e = f(6);

    // Both calls of get_symbol("x") yield the same symbol.
    cout << e.degree(get_symbol("x")) << endl;
     // prints "6"
}
```

Instead of creating symbols from strings we could also have `get_symbol()` take, for example, an integer number as its argument. In this case, we would probably want to give the generated symbols names that include this number, which can be accomplished with the help of an `ostringstream`.

In general, if you're getting weird results from GiNaC such as an expression '`x-x`' that is not simplified to zero, you should check your symbol definitions.

As we said, the names of symbols primarily serve for purposes of expression output. But there are actually two instances where GiNaC uses the names for identifying symbols: When constructing an expression from a string, and when recreating an expression from an archive (see Section 5.15 [Input/output], page 81).

In addition to its name, a symbol may contain a special string that is used in LaTeX output:

```
symbol x("x", "\\Box");
```

This creates a symbol that is printed as `"x"` in normal output, but as `"\Box"` in LaTeX code (See Section 5.15 [Input/output], page 81, for more information about the different output formats of expressions in GiNaC). GiNaC automatically creates proper LaTeX code for symbols having names of greek letters ('`alpha`', '`mu`', etc.). You can retrieve the name and the LaTeX name of a symbol using the respective methods:

```
symbol::get_name() const;
symbol::get_TeX_name() const;
```

Symbols in GiNaC can't be assigned values. If you need to store results of calculations and give them a name, use C++ variables of type `ex`. If you want to replace a symbol in an expression with something else, you can invoke the expression's `.subs()` method (see Section 5.3 [Substituting expressions], page 53).

By default, symbols are expected to stand in for complex values, i.e. they live in the complex domain. As a consequence, operations like complex conjugation, for example (see Section 5.13 [Complex expressions], page 80), do *not* evaluate if applied to such symbols. Likewise `log(exp(x))` does not evaluate to `x`, because of the unknown imaginary part of `x`. On the other

hand, if you are sure that your symbols will hold only real values, you would like to have such functions evaluated. Therefore GiNaC allows you to specify the domain of the symbol. Instead of `symbol x("x");` you can write `realsymbol x("x");` to tell GiNaC that `x` stands in for real values.

Furthermore, it is also possible to declare a symbol as positive. This will, for instance, enable the automatic simplification of `abs(x)` into `x`. This is done by declaring the symbol as `possymbol x("x");`.

4.6 Numbers

For storing numerical things, GiNaC uses Bruno Haible's library CLN. The classes therein serve as foundation classes for GiNaC. CLN stands for Class Library for Numbers or alternatively for Common Lisp Numbers. In order to find out more about CLN's internals, the reader is referred to the documentation of that library. See *The CLN Manual*, for more information. Suffice to say that it is by itself build on top of another library, the GNU Multiple Precision library GMP, which is an extremely fast library for arbitrary long integers and rationals as well as arbitrary precision floating point numbers. It is very commonly used by several popular cryptographic applications. CLN extends GMP by several useful things: First, it introduces the complex number field over either reals (i.e. floating point numbers with arbitrary precision) or rationals. Second, it automatically converts rationals to integers if the denominator is unity and complex numbers to real numbers if the imaginary part vanishes and also correctly treats algebraic functions. Third it provides good implementations of state-of-the-art algorithms for all trigonometric and hyperbolic functions as well as for calculation of some useful constants.

The user can construct an object of class `numeric` in several ways. The following example shows the four most important constructors. It uses construction from C-integer, construction of fractions from two integers, construction from C-float and construction from a string:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace GiNaC;

int main()
{
    numeric two = 2;                     // exact integer 2
    numeric r(2,3);                      // exact fraction 2/3
    numeric e(2.71828);                  // floating point number
    numeric p = "3.14159265358979323846"; // constructor from string
    // Trott's constant in scientific notation:
    numeric trott("1.0841015122311136151E-2");

    std::cout << two*p << std::endl;  // floating point 6.283...
    ...
```

The imaginary unit in GiNaC is a predefined `numeric` object with the name `I`:

```
    ...
    numeric z1 = 2-3*I;                  // exact complex number 2-3i
    numeric z2 = 5.9+1.6*I;              // complex floating point number
}
```

It may be tempting to construct fractions by writing `numeric r(3/2)`. This would, however, call C's built-in operator `/` for integers first and result in a numeric holding a plain integer 1. **Never use the operator / on integers** unless you know exactly what you are doing! Use the constructor from two integers instead, as shown in the example above. Writing `numeric(1)/2` may look funny but works also.

We have seen now the distinction between exact numbers and floating point numbers. Clearly, the user should never have to worry about dynamically created exact numbers, since their 'exactness' always determines how they ought to be handled, i.e. how 'long' they are. The situation is different for floating point numbers. Their accuracy is controlled by one *global* variable, called `Digits`. (For those readers who know about Maple: it behaves very much like Maple's `Digits`). All objects of class numeric that are constructed from then on will be stored with a precision matching that number of decimal digits:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

void foo()
{
    numeric three(3.0), one(1.0);
    numeric x = one/three;

    cout << "in " << Digits << " digits:" << endl;
    cout << x << endl;
    cout << Pi.evalf() << endl;
}

int main()
{
    foo();
    Digits = 60;
    foo();
    return 0;
}
```

The above example prints the following output to screen:

```
in 17 digits:
0.33333333333333333
3.1415926535897932385
in 60 digits:
0.333333333333333333333333333333333333333333333333333333333333333
3.14159265358979323846264338327950288419716939937510582097494459230 78
```

Note that the last number is not necessarily rounded as you would naively expect it to be rounded in the decimal system. But note also, that in both cases you got a couple of extra digits. This is because numbers are internally stored by CLN as chunks of binary digits in order to match your machine's word size and to not waste precision. Thus, on architectures with different word size, the above output might even differ with regard to actually computed digits.

It should be clear that objects of class `numeric` should be used for constructing numbers or for doing arithmetic with them. The objects one deals with most of the time are the polymorphic expressions `ex`.

4.6.1 Tests on numbers

Once you have declared some numbers, assigned them to expressions and done some arithmetic with them it is frequently desired to retrieve some kind of information from them like asking whether that number is integer, rational, real or complex. For those cases GiNaC provides several useful methods. (Internally, they fall back to invocations of certain CLN functions.)

As an example, let's construct some rational number, multiply it with some multiple of its denominator and test what comes out:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

// some very important constants:
const numeric twentyone(21);
const numeric ten(10);
const numeric five(5);

int main()
{
    numeric answer = twentyone;

    answer /= five;
    cout << answer.is_integer() << endl;  // false, it's 21/5
    answer *= ten;
    cout << answer.is_integer() << endl;  // true, it's 42 now!
}
```

Note that the variable `answer` is constructed here as an integer by `numeric`'s copy constructor, but in an intermediate step it holds a rational number represented as integer numerator and integer denominator. When multiplied by 10, the denominator becomes unity and the result is automatically converted to a pure integer again. Internally, the underlying CLN is responsible for this behavior and we refer the reader to CLN's documentation. Suffice to say that the same behavior applies to complex numbers as well as return values of certain functions. Complex numbers are automatically converted to real numbers if the imaginary part becomes zero. The full set of tests that can be applied is listed in the following table.

Method	**Returns true if the object is...**
`.is_zero()`	...equal to zero
`.is_positive()`	...not complex and greater than 0
`.is_negative()`	...not complex and smaller than 0
`.is_integer()`	...a (non-complex) integer
`.is_pos_integer()`	...an integer and greater than 0
`.is_nonneg_integer()`	...an integer and greater equal 0
`.is_even()`	...an even integer
`.is_odd()`	...an odd integer
`.is_prime()`	...a prime integer (probabilistic primality test)
`.is_rational()`	...an exact rational number (integers are rational, too)
`.is_real()`	...a real integer, rational or float (i.e. is not complex)
`.is_cinteger()`	...a (complex) integer (such as $2-3*I$)
`.is_crational()`	...an exact (complex) rational number (such as $2/3+7/2*I$)

4.6.2 Numeric functions

The following functions can be applied to `numeric` objects and will be evaluated immediately:

Name	**Function**
`inverse(z)`	returns $1/z$
`pow(a, b)`	exponentiation a^b
`abs(z)`	absolute value
`real(z)`	real part
`imag(z)`	imaginary part
`csgn(z)`	complex sign (returns an `int`)
`step(x)`	step function (returns an `numeric`)
`numer(z)`	numerator of rational or complex rational number
`denom(z)`	denominator of rational or complex rational number
`sqrt(z)`	square root
`isqrt(n)`	integer square root
`sin(z)`	sine
`cos(z)`	cosine
`tan(z)`	tangent
`asin(z)`	inverse sine
`acos(z)`	inverse cosine
`atan(z)`	inverse tangent
`atan(y, x)`	inverse tangent with two arguments
`sinh(z)`	hyperbolic sine
`cosh(z)`	hyperbolic cosine
`tanh(z)`	hyperbolic tangent
`asinh(z)`	inverse hyperbolic sine
`acosh(z)`	inverse hyperbolic cosine
`atanh(z)`	inverse hyperbolic tangent
`exp(z)`	exponential function
`log(z)`	natural logarithm
`Li2(z)`	dilogarithm
`zeta(z)`	Riemann's zeta function
`tgamma(z)`	gamma function
`lgamma(z)`	logarithm of gamma function
`psi(z)`	psi (digamma) function
`psi(n, z)`	derivatives of psi function (polygamma functions)
`factorial(n)`	factorial function $n!$
`doublefactorial(n)`	double factorial function $n!!$
`binomial(n, k)`	binomial coefficients
`bernoulli(n)`	Bernoulli numbers
`fibonacci(n)`	Fibonacci numbers
`mod(a, b)`	modulus in positive representation (in the range `[0, abs(b)-1]` with the sign of b, or zero)
`smod(a, b)`	modulus in symmetric representation (in the range `[-iquo(abs(b), 2), iquo(abs(b), 2)]`)
`irem(a, b)`	integer remainder (has the sign of a, or is zero)
`irem(a, b, q)`	integer remainder and quotient, `irem(a, b, q) == a-q*b`
`iquo(a, b)`	integer quotient
`iquo(a, b, r)`	integer quotient and remainder, `r == a-iquo(a, b)*b`
`gcd(a, b)`	greatest common divisor
`lcm(a, b)`	least common multiple

Most of these functions are also available as symbolic functions that can be used in expressions (see Section 4.10 [Mathematical functions], page 24) or, like `gcd()`, as polynomial algorithms.

4.6.3 Converting numbers

Sometimes it is desirable to convert a `numeric` object back to a built-in arithmetic type (`int`, `double`, etc.). The `numeric` class provides a couple of methods for this purpose:

```
int numeric::to_int() const;
long numeric::to_long() const;
double numeric::to_double() const;
cln::cl_N numeric::to_cl_N() const;
```

`to_int()` and `to_long()` only work when the number they are applied on is an exact integer. Otherwise the program will halt with a message like '`Not a 32-bit integer`'. `to_double()` applied on a rational number will return a floating-point approximation. Both `to_int()/to_long()` and `to_double()` discard the imaginary part of complex numbers.

Note the signature of the above methods, you may need to apply a type conversion and call `evalf()` as shown in the following example:

```
...
ex e1 = 1, e2 = sin(Pi/5);
cout << ex_to<numeric>(e1).to_int() << endl
     << ex_to<numeric>(e2.evalf()).to_double() << endl;
...
```

4.7 Constants

Constants behave pretty much like symbols except that they return some specific number when the method `.evalf()` is called.

The predefined known constants are:

Name	**Common Name**	**Numerical Value (to 35 digits)**
`Pi`	Archimedes' constant	3.14159265358979323846264338327950288
`Catalan`	Catalan's constant	0.91596559417721901505460351493238411
`Euler`	Euler's (or Euler-Mascheroni) constant	0.57721566490153286060651209008240243

4.8 Sums, products and powers

Simple rational expressions are written down in GiNaC pretty much like in other CAS or like expressions involving numerical variables in C. The necessary operators `+`, `-`, `*` and `/` have been overloaded to achieve this goal. When you run the following code snippet, the constructor for an object of type `mul` is automatically called to hold the product of `a` and `b` and then the constructor for an object of type `add` is called to hold the sum of that `mul` object and the number one:

```
...
symbol a("a"), b("b");
ex MyTerm = 1+a*b;
...
```

For exponentiation, you have already seen the somewhat clumsy (though C-ish) statement `pow(x,2);` to represent `x` squared. This direct construction is necessary since we cannot safely overload the constructor `^` in C++ to construct a `power` object. If we did, it would have several counterintuitive and undesired effects:

- Due to C's operator precedence, `2*x^2` would be parsed as `(2*x)^2`.

- Due to the binding of the operator `^`, `x^a^b` would result in `(x^a)^b`. This would be confusing since most (though not all) other CAS interpret this as `x^(a^b)`.
- Also, expressions involving integer exponents are very frequently used, which makes it even more dangerous to overload `^` since it is then hard to distinguish between the semantics as exponentiation and the one for exclusive or. (It would be embarrassing to return `1` where one has requested `2^3`.)

All effects are contrary to mathematical notation and differ from the way most other CAS handle exponentiation, therefore overloading `^` is ruled out for GiNaC's C++ part. The situation is different in `ginsh`, there the exponentiation-`^` exists. (Also note that the other frequently used exponentiation operator `**` does not exist at all in C++).

To be somewhat more precise, objects of the three classes described here, are all containers for other expressions. An object of class `power` is best viewed as a container with two slots, one for the basis, one for the exponent. All valid GiNaC expressions can be inserted. However, basic transformations like simplifying `pow(pow(x,2),3)` to `x^6` automatically are only performed when this is mathematically possible. If we replace the outer exponent three in the example by some symbols `a`, the simplification is not safe and will not be performed, since `a` might be `1/2` and `x` negative.

Objects of type `add` and `mul` are containers with an arbitrary number of slots for expressions to be inserted. Again, simple and safe simplifications are carried out like transforming `3*x+4-x` to `2*x+4`.

4.9 Lists of expressions

The GiNaC class `lst` serves for holding a *list* of arbitrary expressions. They are not as ubiquitous as in many other computer algebra packages, but are sometimes used to supply a variable number of arguments of the same type to GiNaC methods such as `subs()` and some `matrix` constructors, so you should have a basic understanding of them.

Lists can be constructed from an initializer list of expressions:

```
{
    symbol x("x"), y("y");
    lst l = {x, 2, y, x+y};
    // now, l is a list holding the expressions 'x', '2', 'y', and 'x+y',
    // in that order
    ...
```

Use the `nops()` method to determine the size (number of expressions) of a list and the `op()` method or the `[]` operator to access individual elements:

```
    ...
    cout << l.nops() << endl;                // prints '4'
    cout << l.op(2) << " " << l[0] << endl;  // prints 'y x'
    ...
```

As with the standard `list<T>` container, accessing random elements of a `lst` is generally an operation of order $O(N)$. Faster read-only sequential access to the elements of a list is possible with the iterator types provided by the `lst` class:

```
typedef ... lst::const_iterator;
typedef ... lst::const_reverse_iterator;
lst::const_iterator lst::begin() const;
lst::const_iterator lst::end() const;
lst::const_reverse_iterator lst::rbegin() const;
lst::const_reverse_iterator lst::rend() const;
```

For example, to print the elements of a list individually you can use:

```
...
// O(N)
for (lst::const_iterator i = l.begin(); i != l.end(); ++i)
    cout << *i << endl;
...
```

which is one order faster than

```
...
// O(N^2)
for (size_t i = 0; i < l.nops(); ++i)
    cout << l.op(i) << endl;
...
```

These iterators also allow you to use some of the algorithms provided by the C++ standard library:

```
...
// print the elements of the list using ranged-based for loop
for (auto i: l)
    cout << i << endl;

// sum up the elements of the list (requires #include <numeric>)
ex sum = std::accumulate(l.begin(), l.end(), ex(0));
cout << sum << endl;  // prints '2+2*x+2*y'
...
```

`lst` is one of the few GiNaC classes that allow in-place modifications (the only other one is `matrix`). You can modify single elements:

```
...
l[1] = 42;       // l is now {x, 42, y, x+y}
l.let_op(1) = 7; // l is now {x, 7, y, x+y}
...
```

You can append or prepend an expression to a list with the `append()` and `prepend()` methods:

```
...
l.append(4*x);   // l is now {x, 7, y, x+y, 4*x}
l.prepend(0);    // l is now {0, x, 7, y, x+y, 4*x}
...
```

You can remove the first or last element of a list with `remove_first()` and `remove_last()`:

```
...
l.remove_first();   // l is now {x, 7, y, x+y, 4*x}
l.remove_last();    // l is now {x, 7, y, x+y}
...
```

You can remove all the elements of a list with `remove_all()`:

```
...
l.remove_all();     // l is now empty
...
```

You can bring the elements of a list into a canonical order with `sort()`:

```
...
lst l1 = {x, 2, y, x+y};
lst l2 = {2, x+y, x, y};
l1.sort();
```

```
        l2.sort();
        // l1 and l2 are now equal
        ...
```

Finally, you can remove all but the first element of consecutive groups of elements with `unique()`:

```
        ...
        lst l3 = {x, 2, 2, 2, y, x+y, y+x};
        l3.unique();        // l3 is now {x, 2, y, x+y}
    }
```

4.10 Mathematical functions

There are quite a number of useful functions hard-wired into GiNaC. For instance, all trigonometric and hyperbolic functions are implemented (See Section 5.12 [Built-in functions], page 74, for a complete list).

These functions (better called *pseudofunctions*) are all objects of class `function`. They accept one or more expressions as arguments and return one expression. If the arguments are not numerical, the evaluation of the function may be halted, as it does in the next example, showing how a function returns itself twice and finally an expression that may be really useful:

```
        ...
        symbol x("x"), y("y");
        ex foo = x+y/2;
        cout << tgamma(foo) << endl;
         // -> tgamma(x+(1/2)*y)
        ex bar = foo.subs(y==1);
        cout << tgamma(bar) << endl;
         // -> tgamma(x+1/2)
        ex foobar = bar.subs(x==7);
        cout << tgamma(foobar) << endl;
         // -> (135135/128)*Pi^(1/2)
        ...
```

Besides evaluation most of these functions allow differentiation, series expansion and so on. Read the next chapter in order to learn more about this.

It must be noted that these pseudofunctions are created by inline functions, where the argument list is templated. This means that whenever you call `GiNaC::sin(1)` it is equivalent to `sin(ex(1))` and will therefore not result in a floating point number. Unless of course the function prototype is explicitly overridden – which is the case for arguments of type `numeric` (not wrapped inside an `ex`). Hence, in order to obtain a floating point number of class `numeric` you should call `sin(numeric(1))`. This is almost the same as calling `sin(1).evalf()` except that the latter will return a numeric wrapped inside an `ex`.

4.11 Relations

Sometimes, a relation holding between two expressions must be stored somehow. The class `relational` is a convenient container for such purposes. A relation is by definition a container for two `ex` and a relation between them that signals equality, inequality and so on. They are created by simply using the C++ operators `==`, `!=`, `<`, `<=`, `>` and `>=` between two expressions.

See Section 4.10 [Mathematical functions], page 24, for examples where various applications of the `.subs()` method show how objects of class relational are used as arguments. There they provide an intuitive syntax for substitutions. They are also used as arguments to the `ex::series` method, where the left hand side of the relation specifies the variable to expand in and the right

hand side the expansion point. They can also be used for creating systems of equations that are to be solved for unknown variables.

But the most common usage of objects of this class is rather inconspicuous in statements of the form `if (expand(pow(a+b,2))==a*a+2*a*b+b*b) {...}`. Here, an implicit conversion from `relational` to `bool` takes place. Note, however, that `==` here does not perform any simplifications, hence `expand()` must be called explicitly.

Simplifications of relationals may be more efficient if preceded by a call to

```
ex relational::canonical() const
```

which returns an equivalent relation with the zero right-hand side. For example:

```
possymbol p("p");
relational rel = (p >= (p*p-1)/p);
if (ex_to<relational>(rel.canonical().normal()))
cout << "correct inequality" << endl;
```

However, a user shall not expect that any inequality can be fully resolved by GiNaC.

4.12 Integrals

An object of class *integral* can be used to hold a symbolic integral. If you want to symbolically represent the integral of `x*x` from 0 to 1, you would write this as

```
integral(x, 0, 1, x*x)
```

The first argument is the integration variable. It should be noted that GiNaC is not very good (yet?) at symbolically evaluating integrals. In fact, it can only integrate polynomials. An expression containing integrals can be evaluated symbolically by calling the

```
.eval_integ()
```

method on it. Numerical evaluation is available by calling the

```
.evalf()
```

method on an expression containing the integral. This will only evaluate integrals into a number if `subs`ing the integration variable by a number in the fourth argument of an integral and then `evalf`ing the result always results in a number. Of course, also the boundaries of the integration domain must `evalf` into numbers. It should be noted that trying to `evalf` a function with discontinuities in the integration domain is not recommended. The accuracy of the numeric evaluation of integrals is determined by the static member variable

```
ex integral::relative_integration_error
```

of the class `integral`. The default value of this is 10^-8. The integration works by halving the interval of integration, until numeric stability of the answer indicates that the requested accuracy has been reached. The maximum depth of the halving can be set via the static member variable

```
int integral::max_integration_level
```

The default value is 15. If this depth is exceeded, `evalf` will simply return the integral unevaluated. The function that performs the numerical evaluation, is also available as

```
ex adaptivesimpson(const ex & x, const ex & a, const ex & b, const ex & f,
                   const ex & error)
```

This function will throw an exception if the maximum depth is exceeded. The last parameter of the function is optional and defaults to the `relative_integration_error`. To make sure that we do not do too much work if an expression contains the same integral multiple times, a lookup table is used.

If you know that an expression holds an integral, you can get the integration variable, the left boundary, right boundary and integrand by respectively calling `.op(0)`, `.op(1)`, `.op(2)`, and `.op(3)`. Differentiating integrals with respect to variables works as expected. Note that it makes no sense to differentiate an integral with respect to the integration variable.

4.13 Matrices

A *matrix* is a two-dimensional array of expressions. The elements of a matrix with m rows and n columns are accessed with two `unsigned` indices, the first one in the range $0 \ldots m-1$, the second one in the range $0 \ldots n-1$.

There are a couple of ways to construct matrices, with or without preset elements. The constructor

```
matrix::matrix(unsigned r, unsigned c);
```

creates a matrix with '`r`' rows and '`c`' columns with all elements set to zero.

The easiest way to create a matrix is using an initializer list of initializer lists, all of the same size:

```
{
    matrix m = {{1, -a},
                {a,  1}};
}
```

You can also specify the elements as a (flat) list with

```
matrix::matrix(unsigned r, unsigned c, const lst & l);
```

The function

```
ex lst_to_matrix(const lst & l);
```

constructs a matrix from a list of lists, each list representing a matrix row.

There is also a set of functions for creating some special types of matrices:

```
ex diag_matrix(const lst & l);
ex diag_matrix(initializer_list<ex> l);
ex unit_matrix(unsigned x);
ex unit_matrix(unsigned r, unsigned c);
ex symbolic_matrix(unsigned r, unsigned c, const string & base_name);
ex symbolic_matrix(unsigned r, unsigned c, const string & base_name,
                   const string & tex_base_name);
```

`diag_matrix()` constructs a square diagonal matrix given the diagonal elements. `unit_matrix()` creates an '`x`' by '`x`' (or '`r`' by '`c`') unit matrix. And finally, `symbolic_matrix` constructs a matrix filled with newly generated symbols made of the specified base name and the position of each element in the matrix.

Matrices often arise by omitting elements of another matrix. For instance, the submatrix `S` of a matrix `M` takes a rectangular block from `M`. The reduced matrix `R` is defined by removing one row and one column from a matrix `M`. (The determinant of a reduced matrix is called a *Minor* of `M` and can be used for computing the inverse using Cramer's rule.)

```
ex sub_matrix(const matrix&m, unsigned r, unsigned nr, unsigned c, unsigned nc);
ex reduced_matrix(const matrix& m, unsigned r, unsigned c);
```

The function `sub_matrix()` takes a row offset `r` and a column offset `c` and takes a block of `nr` rows and `nc` columns. The function `reduced_matrix()` has two integer arguments that specify which row and column to remove:

```
{
    matrix m = {{11, 12, 13},
                {21, 22, 23},
                {31, 32, 33}};
    cout << reduced_matrix(m, 1, 1) << endl;
    // -> [[11,13],[31,33]]
    cout << sub_matrix(m, 1, 2, 1, 2) << endl;
    // -> [[22,23],[32,33]]
```

```
}
```

Matrix elements can be accessed and set using the parenthesis (function call) operator:

```
const ex & matrix::operator()(unsigned r, unsigned c) const;
ex & matrix::operator()(unsigned r, unsigned c);
```

It is also possible to access the matrix elements in a linear fashion with the `op()` method. But C++-style subscripting with square brackets '`[]`' is not available.

Here are a couple of examples for constructing matrices:

```
{
    symbol a("a"), b("b");

    matrix M = {{a, 0},
                {0, b}};
    cout << M << endl;
     // -> [[a,0],[0,b]]

    matrix M2(2, 2);
    M2(0, 0) = a;
    M2(1, 1) = b;
    cout << M2 << endl;
     // -> [[a,0],[0,b]]

    cout << matrix(2, 2, lst{a, 0, 0, b}) << endl;
     // -> [[a,0],[0,b]]

    cout << lst_to_matrix(lst{lst{a, 0}, lst{0, b}}) << endl;
     // -> [[a,0],[0,b]]

    cout << diag_matrix(lst{a, b}) << endl;
     // -> [[a,0],[0,b]]

    cout << unit_matrix(3) << endl;
     // -> [[1,0,0],[0,1,0],[0,0,1]]

    cout << symbolic_matrix(2, 3, "x") << endl;
     // -> [[x00,x01,x02],[x10,x11,x12]]
}
```

The method `matrix::is_zero_matrix()` returns `true` only if all entries of the matrix are zeros. There is also method `ex::is_zero_matrix()` which returns `true` only if the expression is zero or a zero matrix.

There are three ways to do arithmetic with matrices. The first (and most direct one) is to use the methods provided by the `matrix` class:

```
matrix matrix::add(const matrix & other) const;
matrix matrix::sub(const matrix & other) const;
matrix matrix::mul(const matrix & other) const;
matrix matrix::mul_scalar(const ex & other) const;
matrix matrix::pow(const ex & expn) const;
matrix matrix::transpose() const;
```

All of these methods return the result as a new matrix object. Here is an example that calculates $A * B - 2 * C$ for three matrices A, B and C:

```
{
```

```
    matrix A = {{ 1, 2},
                { 3, 4}};
    matrix B = {{-1, 0},
                { 2, 1}};
    matrix C = {{ 8, 4},
                { 2, 1}};

    matrix result = A.mul(B).sub(C.mul_scalar(2));
    cout << result << endl;
     // -> [[-13,-6],[1,2]]
    ...
}
```

The second (and probably the most natural) way is to construct an expression containing matrices with the usual arithmetic operators and `pow()`. For efficiency reasons, expressions with sums, products and powers of matrices are not automatically evaluated in GiNaC. You have to call the method

```
ex ex::evalm() const;
```

to obtain the result:

```
{
    ...
    ex e = A*B - 2*C;
    cout << e << endl;
     // -> [[1,2],[3,4]]*[[-1,0],[2,1]]-2*[[8,4],[2,1]]
    cout << e.evalm() << endl;
     // -> [[-13,-6],[1,2]]
    ...
}
```

The non-commutativity of the product `A*B` in this example is automatically recognized by GiNaC. There is no need to use a special operator here. See Section 4.15 [Non-commutative objects], page 39, for more information about dealing with non-commutative expressions.

Finally, you can work with indexed matrices and call `simplify_indexed()` to perform the arithmetic:

```
{
    ...
    idx i(symbol("i"), 2), j(symbol("j"), 2), k(symbol("k"), 2);
    e = indexed(A, i, k) * indexed(B, k, j) - 2 * indexed(C, i, j);
    cout << e << endl;
     // -> -2*[[8,4],[2,1]].i.j+[[-1,0],[2,1]].k.j*[[1,2],[3,4]].i.k
    cout << e.simplify_indexed() << endl;
     // -> [[-13,-6],[1,2]].i.j
}
```

Using indices is most useful when working with rectangular matrices and one-dimensional vectors because you don't have to worry about having to transpose matrices before multiplying them. See Section 4.14 [Indexed objects], page 29, for more information about using matrices with indices, and about indices in general.

The `matrix` class provides a couple of additional methods for computing determinants, traces, characteristic polynomials and ranks:

```
ex matrix::determinant(unsigned algo=determinant_algo::automatic) const;
ex matrix::trace() const;
ex matrix::charpoly(const ex & lambda) const;
```

```
unsigned matrix::rank(unsigned algo=solve_algo::automatic) const;
```

The optional '`algo`' argument of `determinant()` and `rank()` functions allows to select between different algorithms for calculating the determinant and rank respectively. The asymptotic speed (as parametrized by the matrix size) can greatly differ between those algorithms, depending on the nature of the matrix' entries. The possible values are defined in the `flags.h` header file. By default, GiNaC uses a heuristic to automatically select an algorithm that is likely (but not guaranteed) to give the result most quickly.

Linear systems can be solved with:

```
matrix matrix::solve(const matrix & vars, const matrix & rhs,
                     unsigned algo=solve_algo::automatic) const;
```

Assuming the matrix object this method is applied on is an `m` times `n` matrix, then `vars` must be a `n` times `p` matrix of symbolic indeterminates and `rhs` a `m` times `p` matrix. The returned matrix then has dimension `n` times `p` and in the case of an underdetermined system will still contain some of the indeterminates from `vars`. If the system is overdetermined, an exception is thrown.

To invert a matrix, use the method:

```
matrix matrix::inverse(unsigned algo=solve_algo::automatic) const;
```

The '`algo`' argument is optional. If given, it must be one of `solve_algo` defined in `flags.h`.

4.14 Indexed objects

GiNaC allows you to handle expressions containing general indexed objects in arbitrary spaces. It is also able to canonicalize and simplify such expressions and perform symbolic dummy index summations. There are a number of predefined indexed objects provided, like delta and metric tensors.

There are few restrictions placed on indexed objects and their indices and it is easy to construct nonsense expressions, but our intention is to provide a general framework that allows you to implement algorithms with indexed quantities, getting in the way as little as possible.

4.14.1 Indexed quantities and their indices

Indexed expressions in GiNaC are constructed of two special types of objects, *index objects* and *indexed objects*.

- Index objects are of class `idx` or a subclass. Every index has a *value* and a *dimension* (which is the dimension of the space the index lives in) which can both be arbitrary expressions but are usually a number or a simple symbol. In addition, indices of class `varidx` have a *variance* (they can be co- or contravariant), and indices of class `spinidx` have a variance and can be *dotted* or *undotted*.
- Indexed objects are of class `indexed` or a subclass. They contain a *base expression* (which is the expression being indexed), and one or more indices.

Please notice: when printing expressions, covariant indices and indices without variance are denoted '`.i`' while contravariant indices are denoted '`~i`'. Dotted indices have a '`*`' in front of the index value. In the following, we are going to use that notation in the text so instead of $A^i_j k$ we will write '`A~i.j.k`'. Index dimensions are not visible in the output.

A simple example shall illustrate the concepts:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;
```

```
int main()
{
    symbol i_sym("i"), j_sym("j");
    idx i(i_sym, 3), j(j_sym, 3);

    symbol A("A");
    cout << indexed(A, i, j) << endl;
     // -> A.i.j
    cout << index_dimensions << indexed(A, i, j) << endl;
     // -> A.i[3].j[3]
    cout << dflt; // reset cout to default output format (dimensions hidden)
    ...
```

The `idx` constructor takes two arguments, the index value and the index dimension. First we define two index objects, `i` and `j`, both with the numeric dimension 3. The value of the index `i` is the symbol `i_sym` (which prints as '`i`') and the value of the index `j` is the symbol `j_sym` (which prints as '`j`'). Next we construct an expression containing one indexed object, '`A.i.j`'. It has the symbol `A` as its base expression and the two indices `i` and `j`.

The dimensions of indices are normally not visible in the output, but one can request them to be printed with the `index_dimensions` manipulator, as shown above.

Note the difference between the indices `i` and `j` which are of class `idx`, and the index values which are the symbols `i_sym` and `j_sym`. The indices of indexed objects cannot directly be symbols or numbers but must be index objects. For example, the following is not correct and will raise an exception:

```
symbol i("i"), j("j");
e = indexed(A, i, j); // ERROR: indices must be of type idx
```

You can have multiple indexed objects in an expression, index values can be numeric, and index dimensions symbolic:

```
    ...
    symbol B("B"), dim("dim");
    cout << 4 * indexed(A, i)
          + indexed(B, idx(j_sym, 4), idx(2, 3), idx(i_sym, dim)) << endl;
     // -> B.j.2.i+4*A.i
    ...
```

`B` has a 4-dimensional symbolic index '`k`', a 3-dimensional numeric index of value 2, and a symbolic index '`i`' with the symbolic dimension '`dim`'. Note that GiNaC doesn't automatically notify you that the free indices of '`A`' and '`B`' in the sum don't match (you have to call `simplify_indexed()` for that, see below).

In fact, base expressions, index values and index dimensions can be arbitrary expressions:

```
    ...
    cout << indexed(A+B, idx(2*i_sym+1, dim/2)) << endl;
     // -> (B+A).(1+2*i)
    ...
```

It's also possible to construct nonsense like '`Pi.sin(x)`'. You will not get an error message from this but you will probably not be able to do anything useful with it.

The methods

```
ex idx::get_value();
ex idx::get_dim();
```

return the value and dimension of an `idx` object. If you have an index in an expression, such as returned by calling `.op()` on an indexed object, you can get a reference to the `idx` object with the function `ex_to<idx>()` on the expression.

There are also the methods

```
bool idx::is_numeric();
bool idx::is_symbolic();
bool idx::is_dim_numeric();
bool idx::is_dim_symbolic();
```

for checking whether the value and dimension are numeric or symbolic (non-numeric). Using the `info()` method of an index (see Section 5.1 [Information about expressions], page 48) returns information about the index value.

If you need co- and contravariant indices, use the `varidx` class:

```
...
symbol mu_sym("mu"), nu_sym("nu");
varidx mu(mu_sym, 4), nu(nu_sym, 4); // default is contravariant ~mu, ~nu
varidx mu_co(mu_sym, 4, true);       // covariant index .mu

cout << indexed(A, mu, nu) << endl;
 // -> A~mu~nu
cout << indexed(A, mu_co, nu) << endl;
 // -> A.mu~nu
cout << indexed(A, mu.toggle_variance(), nu) << endl;
 // -> A.mu~nu
...
```

A `varidx` is an `idx` with an additional flag that marks it as co- or contravariant. The default is a contravariant (upper) index, but this can be overridden by supplying a third argument to the `varidx` constructor. The two methods

```
bool varidx::is_covariant();
bool varidx::is_contravariant();
```

allow you to check the variance of a `varidx` object (use `ex_to<varidx>()` to get the object reference from an expression). There's also the very useful method

```
ex varidx::toggle_variance();
```

which makes a new index with the same value and dimension but the opposite variance. By using it you only have to define the index once.

The `spinidx` class provides dotted and undotted variant indices, as used in the Weyl-van-der-Waerden spinor formalism:

```
...
symbol K("K"), C_sym("C"), D_sym("D");
spinidx C(C_sym, 2), D(D_sym);          // default is 2-dimensional,
                                        // contravariant, undotted
spinidx C_co(C_sym, 2, true);           // covariant index
spinidx D_dot(D_sym, 2, false, true);   // contravariant, dotted
spinidx D_co_dot(D_sym, 2, true, true); // covariant, dotted

cout << indexed(K, C, D) << endl;
 // -> K~C~D
cout << indexed(K, C_co, D_dot) << endl;
 // -> K.C~*D
cout << indexed(K, D_co_dot, D) << endl;
 // -> K.*D~D
...
```

A `spinidx` is a `varidx` with an additional flag that marks it as dotted or undotted. The default is undotted but this can be overridden by supplying a fourth argument to the `spinidx` constructor. The two methods

```
bool spinidx::is_dotted();
bool spinidx::is_undotted();
```

allow you to check whether or not a `spinidx` object is dotted (use `ex_to<spinidx>()` to get the object reference from an expression). Finally, the two methods

```
ex spinidx::toggle_dot();
ex spinidx::toggle_variance_dot();
```

create a new index with the same value and dimension but opposite dottedness and the same or opposite variance.

4.14.2 Substituting indices

Sometimes you will want to substitute one symbolic index with another symbolic or numeric index, for example when calculating one specific element of a tensor expression. This is done with the `.subs()` method, as it is done for symbols (see Section 5.3 [Substituting expressions], page 53).

You have two possibilities here. You can either substitute the whole index by another index or expression:

```
...
ex e = indexed(A, mu_co);
cout << e << " becomes " << e.subs(mu_co == nu) << endl;
 // -> A.mu becomes A~nu
cout << e << " becomes " << e.subs(mu_co == varidx(0, 4)) << endl;
 // -> A.mu becomes A~0
cout << e << " becomes " << e.subs(mu_co == 0) << endl;
 // -> A.mu becomes A.0
...
```

The third example shows that trying to replace an index with something that is not an index will substitute the index value instead.

Alternatively, you can substitute the *symbol* of a symbolic index by another expression:

```
...
ex e = indexed(A, mu_co);
cout << e << " becomes " << e.subs(mu_sym == nu_sym) << endl;
 // -> A.mu becomes A.nu
cout << e << " becomes " << e.subs(mu_sym == 0) << endl;
 // -> A.mu becomes A.0
...
```

As you see, with the second method only the value of the index will get substituted. Its other properties, including its dimension, remain unchanged. If you want to change the dimension of an index you have to substitute the whole index by another one with the new dimension.

Finally, substituting the base expression of an indexed object works as expected:

```
...
ex e = indexed(A, mu_co);
cout << e << " becomes " << e.subs(A == A+B) << endl;
 // -> A.mu becomes (B+A).mu
...
```

4.14.3 Symmetries

Indexed objects can have certain symmetry properties with respect to their indices. Symmetries are specified as a tree of objects of class `symmetry` that is constructed with the helper functions

```
symmetry sy_none(...);
symmetry sy_symm(...);
symmetry sy_anti(...);
symmetry sy_cycl(...);
```

`sy_none()` stands for no symmetry, `sy_symm()` and `sy_anti()` specify fully symmetric or antisymmetric, respectively, and `sy_cycl()` represents a cyclic symmetry. Each of these functions accepts up to four arguments which can be either symmetry objects themselves or unsigned integer numbers that represent an index position (counting from 0). A symmetry specification that consists of only a single `sy_symm()`, `sy_anti()` or `sy_cycl()` with no arguments specifies the respective symmetry for all indices.

Here are some examples of symmetry definitions:

```
...
// No symmetry:
e = indexed(A, i, j);
e = indexed(A, sy_none(), i, j);     // equivalent
e = indexed(A, sy_none(0, 1), i, j); // equivalent

// Symmetric in all three indices:
e = indexed(A, sy_symm(), i, j, k);
e = indexed(A, sy_symm(0, 1, 2), i, j, k); // equivalent
e = indexed(A, sy_symm(2, 0, 1), i, j, k); // same symmetry, but yields a
                                           // different canonical order

// Symmetric in the first two indices only:
e = indexed(A, sy_symm(0, 1), i, j, k);
e = indexed(A, sy_none(sy_symm(0, 1), 2), i, j, k); // equivalent

// Antisymmetric in the first and last index only (index ranges need not
// be contiguous):
e = indexed(A, sy_anti(0, 2), i, j, k);
e = indexed(A, sy_none(sy_anti(0, 2), 1), i, j, k); // equivalent

// An example of a mixed symmetry: antisymmetric in the first two and
// last two indices, symmetric when swapping the first and last index
// pairs (like the Riemann curvature tensor):
e = indexed(A, sy_symm(sy_anti(0, 1), sy_anti(2, 3)), i, j, k, l);

// Cyclic symmetry in all three indices:
e = indexed(A, sy_cycl(), i, j, k);
e = indexed(A, sy_cycl(0, 1, 2), i, j, k); // equivalent

// The following examples are invalid constructions that will throw
// an exception at run time.

// An index may not appear multiple times:
e = indexed(A, sy_symm(0, 0, 1), i, j, k); // ERROR
e = indexed(A, sy_none(sy_symm(0, 1), sy_anti(0, 2)), i, j, k); // ERROR
```

```
// Every child of sy_symm(), sy_anti() and sy_cycl() must refer to the
// same number of indices:
e = indexed(A, sy_symm(sy_anti(0, 1), 2), i, j, k); // ERROR

// And of course, you cannot specify indices which are not there:
e = indexed(A, sy_symm(0, 1, 2, 3), i, j, k); // ERROR
...
```

If you need to specify more than four indices, you have to use the `.add()` method of the `symmetry` class. For example, to specify full symmetry in the first six indices you would write `sy_symm(0, 1, 2, 3).add(4).add(5)`.

If an indexed object has a symmetry, GiNaC will automatically bring the indices into a canonical order which allows for some immediate simplifications:

```
...
cout << indexed(A, sy_symm(), i, j)
      + indexed(A, sy_symm(), j, i) << endl;
 // -> 2*A.j.i
cout << indexed(B, sy_anti(), i, j)
      + indexed(B, sy_anti(), j, i) << endl;
 // -> 0
cout << indexed(B, sy_anti(), i, j, k)
      - indexed(B, sy_anti(), j, k, i) << endl;
 // -> 0
...
```

4.14.4 Dummy indices

GiNaC treats certain symbolic index pairs as *dummy indices* meaning that a summation over the index range is implied. Symbolic indices which are not dummy indices are called *free indices*. Numeric indices are neither dummy nor free indices.

To be recognized as a dummy index pair, the two indices must be of the same class and their value must be the same single symbol (an index like '`2*n+1`' is never a dummy index). If the indices are of class `varidx` they must also be of opposite variance; if they are of class `spinidx` they must be both dotted or both undotted.

The method `.get_free_indices()` returns a vector containing the free indices of an expression. It also checks that the free indices of the terms of a sum are consistent:

```
{
    symbol A("A"), B("B"), C("C");

    symbol i_sym("i"), j_sym("j"), k_sym("k"), l_sym("l");
    idx i(i_sym, 3), j(j_sym, 3), k(k_sym, 3), l(l_sym, 3);

    ex e = indexed(A, i, j) * indexed(B, j, k) + indexed(C, k, l, i, l);
    cout << exprseq(e.get_free_indices()) << endl;
     // -> (.i,.k)
     // 'j' and 'l' are dummy indices

    symbol mu_sym("mu"), nu_sym("nu"), rho_sym("rho"), sigma_sym("sigma");
    varidx mu(mu_sym, 4), nu(nu_sym, 4), rho(rho_sym, 4), sigma(sigma_sym, 4);

    e = indexed(A, mu, nu) * indexed(B, nu.toggle_variance(), rho)
      + indexed(C, mu, sigma, rho, sigma.toggle_variance());
```

```
    cout << exprseq(e.get_free_indices()) << endl;
     // -> (~mu,~rho)
     // 'nu' is a dummy index, but 'sigma' is not

    e = indexed(A, mu, mu);
    cout << exprseq(e.get_free_indices()) << endl;
     // -> (~mu)
     // 'mu' is not a dummy index because it appears twice with the same
     // variance

    e = indexed(A, mu, nu) + 42;
    cout << exprseq(e.get_free_indices()) << endl; // ERROR
     // this will throw an exception:
     // "add::get_free_indices: inconsistent indices in sum"
}
```

A dummy index summation like $a_i b^i$ can be expanded for indices with numeric dimensions (e.g. 3) into the explicit sum like $a_1 b^1 + a_2 b^2 + a_3 b^3$. This is performed by the function

```
ex expand_dummy_sum(const ex & e, bool subs_idx = false);
```

which takes an expression `e` and returns the expanded sum for all dummy indices with numeric dimensions. If the parameter `subs_idx` is set to `true` then all substitutions are made by `idx` class indices, i.e. without variance. In this case the above sum $a_i b^i$ will be expanded to $a_1 b_1 + a_2 b_2 + a_3 b_3$.

4.14.5 Simplifying indexed expressions

In addition to the few automatic simplifications that GiNaC performs on indexed expressions (such as re-ordering the indices of symmetric tensors and calculating traces and convolutions of matrices and predefined tensors) there is the method

```
ex ex::simplify_indexed();
ex ex::simplify_indexed(const scalar_products & sp);
```

that performs some more expensive operations:

- it checks the consistency of free indices in sums in the same way `get_free_indices()` does
- it tries to give dummy indices that appear in different terms of a sum the same name to allow simplifications like $a_i * b_i - a_j * b_j = 0$
- it (symbolically) calculates all possible dummy index summations/contractions with the predefined tensors (this will be explained in more detail in the next section)
- it detects contractions that vanish for symmetry reasons, for example the contraction of a symmetric and a totally antisymmetric tensor
- as a special case of dummy index summation, it can replace scalar products of two tensors with a user-defined value

The last point is done with the help of the `scalar_products` class which is used to store scalar products with known values (this is not an arithmetic class, you just pass it to `simplify_indexed()`):

```
{
    symbol A("A"), B("B"), C("C"), i_sym("i");
    idx i(i_sym, 3);

    scalar_products sp;
    sp.add(A, B, 0); // A and B are orthogonal
    sp.add(A, C, 0); // A and C are orthogonal
```

```
    sp.add(A, A, 4); // A^2 = 4 (A has length 2)

    e = indexed(A + B, i) * indexed(A + C, i);
    cout << e << endl;
     // -> (B+A).i*(A+C).i

    cout << e.expand(expand_options::expand_indexed).simplify_indexed(sp)
         << endl;
     // -> 4+C.i*B.i
}
```

The `scalar_products` object `sp` acts as a storage for the scalar products added to it with the `.add()` method. This method takes three arguments: the two expressions of which the scalar product is taken, and the expression to replace it with.

The example above also illustrates a feature of the `expand()` method: if passed the `expand_indexed` option it will distribute indices over sums, so '`(A+B).i`' becomes '`A.i+B.i`'.

4.14.6 Predefined tensors

Some frequently used special tensors such as the delta, epsilon and metric tensors are predefined in GiNaC. They have special properties when contracted with other tensor expressions and some of them have constant matrix representations (they will evaluate to a number when numeric indices are specified).

4.14.6.1 Delta tensor

The delta tensor takes two indices, is symmetric and has the matrix representation `diag(1, 1, 1, ...)`. It is constructed by the function `delta_tensor()`:

```
{
    symbol A("A"), B("B");

    idx i(symbol("i"), 3), j(symbol("j"), 3),
        k(symbol("k"), 3), l(symbol("l"), 3);

    ex e = indexed(A, i, j) * indexed(B, k, l)
         * delta_tensor(i, k) * delta_tensor(j, l);
    cout << e.simplify_indexed() << endl;
     // -> B.i.j*A.i.j

    cout << delta_tensor(i, i) << endl;
     // -> 3
}
```

4.14.6.2 General metric tensor

The function `metric_tensor()` creates a general symmetric metric tensor with two indices that can be used to raise/lower tensor indices. The metric tensor is denoted as '`g`' in the output and if its indices are of mixed variance it is automatically replaced by a delta tensor:

```
{
    symbol A("A");

    varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4), rho(symbol("rho"), 4);

    ex e = metric_tensor(mu, nu) * indexed(A, nu.toggle_variance(), rho);
    cout << e.simplify_indexed() << endl;
```

```
    // -> A~mu~rho

    e = delta_tensor(mu, nu.toggle_variance()) * metric_tensor(nu, rho);
    cout << e.simplify_indexed() << endl;
    // -> g~mu~rho

    e = metric_tensor(mu.toggle_variance(), nu.toggle_variance())
      * metric_tensor(nu, rho);
    cout << e.simplify_indexed() << endl;
    // -> delta.mu~rho

    e = metric_tensor(nu.toggle_variance(), rho.toggle_variance())
      * metric_tensor(mu, nu) * (delta_tensor(mu.toggle_variance(), rho)
        + indexed(A, mu.toggle_variance(), rho));
    cout << e.simplify_indexed() << endl;
    // -> 4+A.rho~rho
}
```

4.14.6.3 Minkowski metric tensor

The Minkowski metric tensor is a special metric tensor with a constant matrix representation which is either `diag(1, -1, -1, ...)` (negative signature, the default) or `diag(-1, 1, 1, ...)` (positive signature). It is created with the function `lorentz_g()` (although it is output as '`eta`'):

```
{
    varidx mu(symbol("mu"), 4);

    e = delta_tensor(varidx(0, 4), mu.toggle_variance())
      * lorentz_g(mu, varidx(0, 4));       // negative signature
    cout << e.simplify_indexed() << endl;
    // -> 1

    e = delta_tensor(varidx(0, 4), mu.toggle_variance())
      * lorentz_g(mu, varidx(0, 4), true); // positive signature
    cout << e.simplify_indexed() << endl;
    // -> -1
}
```

4.14.6.4 Spinor metric tensor

The function `spinor_metric()` creates an antisymmetric tensor with two indices that is used to raise/lower indices of 2-component spinors. It is output as '`eps`':

```
{
    symbol psi("psi");

    spinidx A(symbol("A")), B(symbol("B")), C(symbol("C"));
    ex A_co = A.toggle_variance(), B_co = B.toggle_variance();

    e = spinor_metric(A, B) * indexed(psi, B_co);
    cout << e.simplify_indexed() << endl;
    // -> psi~A

    e = spinor_metric(A, B) * indexed(psi, A_co);
    cout << e.simplify_indexed() << endl;
```

```
    // -> -psi~B

    e = spinor_metric(A_co, B_co) * indexed(psi, B);
    cout << e.simplify_indexed() << endl;
    // -> -psi.A

    e = spinor_metric(A_co, B_co) * indexed(psi, A);
    cout << e.simplify_indexed() << endl;
    // -> psi.B

    e = spinor_metric(A_co, B_co) * spinor_metric(A, B);
    cout << e.simplify_indexed() << endl;
    // -> 2

    e = spinor_metric(A_co, B_co) * spinor_metric(B, C);
    cout << e.simplify_indexed() << endl;
    // -> -delta.A~C
}
```

The matrix representation of the spinor metric is `[[0, 1], [-1, 0]]`.

4.14.6.5 Epsilon tensor

The epsilon tensor is totally antisymmetric, its number of indices is equal to the dimension of the index space (the indices must all be of the same numeric dimension), and '`eps.1.2.3...`' (resp. '`eps~0~1~2...`') is defined to be 1. Its behavior with indices that have a variance also depends on the signature of the metric. Epsilon tensors are output as '`eps`'.

There are three functions defined to create epsilon tensors in 2, 3 and 4 dimensions:

```
ex epsilon_tensor(const ex & i1, const ex & i2);
ex epsilon_tensor(const ex & i1, const ex & i2, const ex & i3);
ex lorentz_eps(const ex & i1, const ex & i2, const ex & i3, const ex & i4,
               bool pos_sig = false);
```

The first two functions create an epsilon tensor in 2 or 3 Euclidean dimensions, the last function creates an epsilon tensor in a 4-dimensional Minkowski space (the last `bool` argument specifies whether the metric has negative or positive signature, as in the case of the Minkowski metric tensor):

```
{
    varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4), rho(symbol("rho"), 4),
           sig(symbol("sig"), 4), lam(symbol("lam"), 4), bet(symbol("bet"), 4);
    e = lorentz_eps(mu, nu, rho, sig) *
        lorentz_eps(mu.toggle_variance(), nu.toggle_variance(), lam, bet);
    cout << simplify_indexed(e) << endl;
    // -> 2*eta~bet~rho*eta~sig~lam-2*eta~sig~bet*eta~rho~lam

    idx i(symbol("i"), 3), j(symbol("j"), 3), k(symbol("k"), 3);
    symbol A("A"), B("B");
    e = epsilon_tensor(i, j, k) * indexed(A, j) * indexed(B, k);
    cout << simplify_indexed(e) << endl;
    // -> -B.k*A.j*eps.i.k.j
    e = epsilon_tensor(i, j, k) * indexed(A, j) * indexed(A, k);
    cout << simplify_indexed(e) << endl;
    // -> 0
}
```

4.14.7 Linear algebra

The `matrix` class can be used with indices to do some simple linear algebra (linear combinations and products of vectors and matrices, traces and scalar products):

```
{
    idx i(symbol("i"), 2), j(symbol("j"), 2);
    symbol x("x"), y("y");

    // A is a 2x2 matrix, X is a 2x1 vector
    matrix A = {{1, 2},
                {3, 4}};
    matrix X = {{x, y}};

    cout << indexed(A, i, i) << endl;
     // -> 5

    ex e = indexed(A, i, j) * indexed(X, j);
    cout << e.simplify_indexed() << endl;
     // -> [[2*y+x],[4*y+3*x]].i

    e = indexed(A, i, j) * indexed(X, i) + indexed(X, j) * 2;
    cout << e.simplify_indexed() << endl;
     // -> [[3*y+3*x,6*y+2*x]].j
}
```

You can of course obtain the same results with the `matrix::add()`, `matrix::mul()` and `matrix::trace()` methods (see Section 4.13 [Matrices], page 26) but with indices you don't have to worry about transposing matrices.

Matrix indices always start at 0 and their dimension must match the number of rows/columns of the matrix. Matrices with one row or one column are vectors and can have one or two indices (it doesn't matter whether it's a row or a column vector). Other matrices must have two indices.

You should be careful when using indices with variance on matrices. GiNaC doesn't look at the variance and doesn't know that '`F~mu~nu`' and '`F.mu.nu`' are different matrices. In this case you should use only one form for '`F`' and explicitly multiply it with a matrix representation of the metric tensor.

4.15 Non-commutative objects

GiNaC is equipped to handle certain non-commutative algebras. Three classes of non-commutative objects are built-in which are mostly of use in high energy physics:

- Clifford (Dirac) algebra (class `clifford`)
- su(3) Lie algebra (class `color`)
- Matrices (unindexed) (class `matrix`)

The `clifford` and `color` classes are subclasses of `indexed` because the elements of these algebras usually carry indices. The `matrix` class is described in more detail in Section 4.13 [Matrices], page 26.

Unlike most computer algebra systems, GiNaC does not primarily provide an operator (often denoted '`&*`') for representing inert products of arbitrary objects. Rather, non-commutativity in GiNaC is a property of the classes of objects involved, and non-commutative products are formed with the usual '`*`' operator, as are ordinary products. GiNaC is capable of figuring out by itself which objects commutate and will group the factors by their class. Consider this example:

```
...
varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4);
idx a(symbol("a"), 8), b(symbol("b"), 8);
ex e = -dirac_gamma(mu) * (2*color_T(a)) * 8 * color_T(b) * dirac_gamma(nu);
cout << e << endl;
 // -> -16*(gamma~mu*gamma~nu)*(T.a*T.b)
...
```

As can be seen, GiNaC pulls out the overall commutative factor '`-16`' and groups the non-commutative factors (the gammas and the su(3) generators) together while preserving the order of factors within each class (because Clifford objects commutate with color objects). The resulting expression is a *commutative* product with two factors that are themselves non-commutative products ('`gamma~mu*gamma~nu`' and '`T.a*T.b`'). For clarification, parentheses are placed around the non-commutative products in the output.

Non-commutative products are internally represented by objects of the class `ncmul`, as opposed to commutative products which are handled by the `mul` class. You will normally not have to worry about this distinction, though.

The advantage of this approach is that you never have to worry about using (or forgetting to use) a special operator when constructing non-commutative expressions. Also, non-commutative products in GiNaC are more intelligent than in other computer algebra systems; they can, for example, automatically canonicalize themselves according to rules specified in the implementation of the non-commutative classes. The drawback is that to work with other than the built-in algebras you have to implement new classes yourself. Both symbols and user-defined functions can be specified as being non-commutative. For symbols, this is done by subclassing class symbol; for functions, by explicitly setting the return type (see Section 6.2 [Symbolic functions], page 91).

Information about the commutativity of an object or expression can be obtained with the two member functions

```
unsigned      ex::return_type() const;
return_type_t ex::return_type_tinfo() const;
```

The `return_type()` function returns one of three values (defined in the header file `flags.h`), corresponding to three categories of expressions in GiNaC:

- `return_types::commutative`: Commutates with everything. Most GiNaC classes are of this kind.
- `return_types::noncommutative`: Non-commutative, belonging to a certain class of non-commutative objects which can be determined with the `return_type_tinfo()` method. Expressions of this category commutate with everything except `noncommutative` expressions of the same class.
- `return_types::noncommutative_composite`: Non-commutative, composed of non-commutative objects of different classes. Expressions of this category don't commutate with any other `noncommutative` or `noncommutative_composite` expressions.

The `return_type_tinfo()` method returns an object of type `return_type_t` that contains information about the type of the expression and, if given, its representation label (see section on dirac gamma matrices for more details). The objects of type `return_type_t` can be tested for equality to test whether two expressions belong to the same category and therefore may not commute.

Here are a couple of examples:

Expression	return_type()
42	commutative
2*x-y	commutative
dirac_ONE()	noncommutative
dirac_gamma(mu)*dirac_gamma(nu)	noncommutative
2*color_T(a)	noncommutative
dirac_ONE()*color_T(a)	noncommutative_composite

A last note: With the exception of matrices, positive integer powers of non-commutative objects are automatically expanded in GiNaC. For example, `pow(a*b, 2)` becomes '`a*b*a*b`' if '`a`' and '`b`' are non-commutative expressions).

4.15.1 Clifford algebra

Clifford algebras are supported in two flavours: Dirac gamma matrices (more physical) and generic Clifford algebras (more mathematical).

4.15.1.1 Dirac gamma matrices

Dirac gamma matrices (note that GiNaC doesn't treat them as matrices) are designated as '`gamma~mu`' and satisfy '`gamma~mu*gamma~nu + gamma~nu*gamma~mu = 2*eta~mu~nu`' where '`eta~mu~nu`' is the Minkowski metric tensor. Dirac gammas are constructed by the function

```
ex dirac_gamma(const ex & mu, unsigned char rl = 0);
```

which takes two arguments: the index and a *representation label* in the range 0 to 255 which is used to distinguish elements of different Clifford algebras (this is also called a *spin line index*). Gammas with different labels commutate with each other. The dimension of the index can be 4 or (in the framework of dimensional regularization) any symbolic value. Spinor indices on Dirac gammas are not supported in GiNaC.

The unity element of a Clifford algebra is constructed by

```
ex dirac_ONE(unsigned char rl = 0);
```

Please notice: You must always use `dirac_ONE()` when referring to multiples of the unity element, even though it's customary to omit it. E.g. instead of `dirac_gamma(mu)*(dirac_slash(q,4)+m)` you have to write `dirac_gamma(mu)*(dirac_slash(q,4)+m*dirac_ONE())`. Otherwise, GiNaC will complain and/or produce incorrect results.

There is a special element '`gamma5`' that commutates with all other gammas, has a unit square, and in 4 dimensions equals '`gamma~0 gamma~1 gamma~2 gamma~3`', provided by

```
ex dirac_gamma5(unsigned char rl = 0);
```

The chiral projectors '`(1+/-gamma5)/2`' are also available as proper objects, constructed by

```
ex dirac_gammaL(unsigned char rl = 0);
ex dirac_gammaR(unsigned char rl = 0);
```

They observe the relations '`gammaL^2 = gammaL`', '`gammaR^2 = gammaR`', and '`gammaL gammaR = gammaR gammaL = 0`'.

Finally, the function

```
ex dirac_slash(const ex & e, const ex & dim, unsigned char rl = 0);
```

creates a term that represents a contraction of '`e`' with the Dirac Lorentz vector (it behaves like a term of the form '`e.mu gamma~mu`' with a unique index whose dimension is given by the `dim` argument). Such slashed expressions are printed with a trailing backslash, e.g. '`e\`'.

In products of dirac gammas, superfluous unity elements are automatically removed, squares are replaced by their values, and '`gamma5`', '`gammaL`' and '`gammaR`' are moved to the front.

The `simplify_indexed()` function performs contractions in gamma strings, for example

```
{
    ...
    symbol a("a"), b("b"), D("D");
    varidx mu(symbol("mu"), D);
    ex e = dirac_gamma(mu) * dirac_slash(a, D)
         * dirac_gamma(mu.toggle_variance());
    cout << e << endl;
     // -> gamma~mu*a\*gamma.mu
    e = e.simplify_indexed();
    cout << e << endl;
     // -> -D*a\+2*a\
    cout << e.subs(D == 4) << endl;
     // -> -2*a\
    ...
}
```

To calculate the trace of an expression containing strings of Dirac gammas you use one of the functions

```
ex dirac_trace(const ex & e, const std::set<unsigned char> & rls,
               const ex & trONE = 4);
ex dirac_trace(const ex & e, const lst & rll, const ex & trONE = 4);
ex dirac_trace(const ex & e, unsigned char rl = 0, const ex & trONE = 4);
```

These functions take the trace over all gammas in the specified set `rls` or list `rll` of representation labels, or the single label `rl`; gammas with other labels are left standing. The last argument to `dirac_trace()` is the value to be returned for the trace of the unity element, which defaults to 4.

The `dirac_trace()` function is a linear functional that is equal to the ordinary matrix trace only in $D = 4$ dimensions. In particular, the functional is not cyclic in $D \neq 4$ dimensions when acting on expressions containing '`gamma5`', so it's not a proper trace. This '`gamma5`' scheme is described in greater detail in the article *The Role of gamma5 in Dimensional Regularization* (Appendix C [Bibliography], page 119).

The value of the trace itself is also usually different in 4 and in $D \neq 4$ dimensions:

```
{
    // 4 dimensions
    varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4), rho(symbol("rho"), 4);
    ex e = dirac_gamma(mu) * dirac_gamma(nu) *
           dirac_gamma(mu.toggle_variance()) * dirac_gamma(rho);
    cout << dirac_trace(e).simplify_indexed() << endl;
     // -> -8*eta~rho~nu
}
...
{
    // D dimensions
    symbol D("D");
    varidx mu(symbol("mu"), D), nu(symbol("nu"), D), rho(symbol("rho"), D);
    ex e = dirac_gamma(mu) * dirac_gamma(nu) *
           dirac_gamma(mu.toggle_variance()) * dirac_gamma(rho);
    cout << dirac_trace(e).simplify_indexed() << endl;
     // -> 8*eta~rho~nu-4*eta~rho~nu*D
}
```

Here is an example for using `dirac_trace()` to compute a value that appears in the calculation of the one-loop vacuum polarization amplitude in QED:

```
{
    symbol q("q"), l("l"), m("m"), ldotq("ldotq"), D("D");
    varidx mu(symbol("mu"), D), nu(symbol("nu"), D);

    scalar_products sp;
    sp.add(l, l, pow(l, 2));
    sp.add(l, q, ldotq);

    ex e = dirac_gamma(mu) *
           (dirac_slash(l, D) + dirac_slash(q, D) + m * dirac_ONE()) *
           dirac_gamma(mu.toggle_variance()) *
           (dirac_slash(l, D) + m * dirac_ONE());
    e = dirac_trace(e).simplify_indexed(sp);
    e = e.collect(lst{l, ldotq, m});
    cout << e << endl;
     // -> (8-4*D)*l^2+(8-4*D)*ldotq+4*D*m^2
}
```

The `canonicalize_clifford()` function reorders all gamma products that appear in an expression to a canonical (but not necessarily simple) form. You can use this to compare two expressions or for further simplifications:

```
{
    varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4);
    ex e = dirac_gamma(mu) * dirac_gamma(nu) + dirac_gamma(nu) * dirac_gamma(mu);
    cout << e << endl;
     // -> gamma~mu*gamma~nu+gamma~nu*gamma~mu

    e = canonicalize_clifford(e);
    cout << e << endl;
     // -> 2*ONE*eta~mu~nu
}
```

4.15.1.2 A generic Clifford algebra

A generic Clifford algebra, i.e. a 2^n dimensional algebra with generators e_k satisfying the identities $e_i e_j + e_j e_i = M(i,j) + M(j,i)$ for some bilinear form (`metric`) $M(i,j)$, which may be non-symmetric (see arXiv:math.QA/9911180) and contain symbolic entries. Such generators are created by the function

```
ex clifford_unit(const ex & mu, const ex & metr, unsigned char rl = 0);
```

where `mu` should be a `idx` (or descendant) class object indexing the generators. Parameter `metr` defines the metric $M(i,j)$ and can be represented by a square `matrix`, `tensormetric` or `indexed` class object. In fact, any expression either with two free indices or without indices at all is admitted as `metr`. In the later case an `indexed` object with two newly created indices with `metr` as its `op(0)` will be used. Optional parameter `rl` allows to distinguish different Clifford algebras, which will commute with each other.

Note that the call `clifford_unit(mu, minkmetric())` creates something very close to `dirac_gamma(mu)`, although `dirac_gamma` have more efficient simplification mechanism. Also, the object created by `clifford_unit(mu, minkmetric())` is not aware about the symmetry of its metric, see the start of the previous paragraph. A more accurate analog of 'dirac_gamma(mu)' should be specifies as follows:

```
clifford_unit(mu, indexed(minkmetric(),sy_symm(),varidx(symbol("i"),4),varidx(symbo
```

The method `clifford::get_metric()` returns a metric defining this Clifford number.

If the matrix $M(i,j)$ is in fact symmetric you may prefer to create the Clifford algebra units with a call like that

```
ex e = clifford_unit(mu, indexed(M, sy_symm(), i, j));
```

since this may yield some further automatic simplifications. Again, for a metric defined through a `matrix` such a symmetry is detected automatically.

Individual generators of a Clifford algebra can be accessed in several ways. For example

```
{
    ...
    idx i(symbol("i"), 4);
    realsymbol s("s");
    ex M = diag_matrix(lst{1, -1, 0, s});
    ex e = clifford_unit(i, M);
    ex e0 = e.subs(i == 0);
    ex e1 = e.subs(i == 1);
    ex e2 = e.subs(i == 2);
    ex e3 = e.subs(i == 3);
    ...
}
```

will produce four anti-commuting generators of a Clifford algebra with properties $e_0^2 = 1$, $e_1^2 = -1$, $e_2^2 = 0$ and $e_3^2 = s$.

A similar effect can be achieved from the function

```
ex lst_to_clifford(const ex & v, const ex & mu,  const ex & metr,
                   unsigned char rl = 0);
ex lst_to_clifford(const ex & v, const ex & e);
```

which converts a list or vector $v = (v^0, v^1, ..., v^n)$ into the Clifford number $v^0 e_0 + v^1 e_1 + ... + v^n e_n$ with '`e.k`' directly supplied in the second form of the procedure. In the first form the Clifford unit '`e.k`' is generated by the call of `clifford_unit(mu, metr, rl)`. If the number of components supplied by `v` exceeds the dimensionality of the Clifford unit `e` by 1 then function `lst_to_clifford()` uses the following pseudo-vector representation: $v^0\mathbf{1} + v^1 e_0 + v^2 e_1 + ... + v^{n+1} e_n$

The previous code may be rewritten with the help of `lst_to_clifford()` as follows

```
{
    ...
    idx i(symbol("i"), 4);
    realsymbol s("s");
    ex M = diag_matrix({1, -1, 0, s});
    ex e0 = lst_to_clifford(lst{1, 0, 0, 0}, i, M);
    ex e1 = lst_to_clifford(lst{0, 1, 0, 0}, i, M);
    ex e2 = lst_to_clifford(lst{0, 0, 1, 0}, i, M);
    ex e3 = lst_to_clifford(lst{0, 0, 0, 1}, i, M);
  ...
}
```

There is the inverse function

```
lst clifford_to_lst(const ex & e, const ex & c, bool algebraic = true);
```

which takes an expression `e` and tries to find a list $v = (v^0, v^1, ..., v^n)$ such that the expression is either vector $e = v^0 c_0 + v^1 c_1 + ... + v^n c_n$ or pseudo-vector $v^0\mathbf{1} + v^1 e_0 + v^2 e_1 + ... + v^{n+1} e_n$ with respect to the given Clifford units `c`. Here none of the '`v~k`' should contain Clifford units

`c` (of course, this may be impossible). This function can use an `algebraic` method (default) or a symbolic one. With the `algebraic` method the '`v~k`' are calculated as $(ec_k + c_k e)/c_k^2$. If c_k^2 is zero or is not `numeric` for some '`k`' then the method will be automatically changed to symbolic. The same effect is obtained by the assignment (`algebraic = false`) in the procedure call.

There are several functions for (anti-)automorphisms of Clifford algebras:

```
ex clifford_prime(const ex & e)
inline ex clifford_star(const ex & e)
inline ex clifford_bar(const ex & e)
```

The automorphism of a Clifford algebra `clifford_prime()` simply changes signs of all Clifford units in the expression. The reversion of a Clifford algebra `clifford_star()` reverses the order of Clifford units in any product. Finally the main anti-automorphism of a Clifford algebra `clifford_bar()` is the composition of the previous two, i.e. it makes the reversion and changes signs of all Clifford units in a product. These functions correspond to the notations e', e^* and $\overline{e}$ used in Clifford algebra textbooks.

The function

```
ex clifford_norm(const ex & e);
```

calculates the norm of a Clifford number from the expression $||e||^2 = e\overline{e}$. The inverse of a Clifford expression is returned by the function

```
ex clifford_inverse(const ex & e);
```

which calculates it as $e^{-1} = \overline{e}/||e||^2$. If $||e|| = 0$ then an exception is raised.

If a Clifford number happens to be a factor of `dirac_ONE()` then we can convert it to a "real" (non-Clifford) expression by the function

```
ex remove_dirac_ONE(const ex & e);
```

The function `canonicalize_clifford()` works for a generic Clifford algebra in a similar way as for Dirac gammas.

The next provided function is

```
ex clifford_moebius_map(const ex & a, const ex & b, const ex & c,
                        const ex & d, const ex & v, const ex & G,
                        unsigned char rl = 0);
ex clifford_moebius_map(const ex & M, const ex & v, const ex & G,
                        unsigned char rl = 0);
```

It takes a list or vector `v` and makes the Moebius (conformal or linear-fractional) transformation '`v -> (av+b)/(cv+d)`' defined by the matrix '`M = [[a, b], [c, d]]`'. The parameter `G` defines the metric of the surrounding (pseudo-)Euclidean space. This can be an indexed object, tensormetric, matrix or a Clifford unit, in the later case the optional parameter `rl` is ignored even if supplied. Depending from the type of `v` the returned value of this function is either a vector or a list holding vector's components.

Finally the function

```
char clifford_max_label(const ex & e, bool ignore_ONE = false);
```

can detect a presence of Clifford objects in the expression `e`: if such objects are found it returns the maximal `representation_label` of them, otherwise `-1`. The optional parameter `ignore_ONE` indicates if `dirac_ONE` objects should be ignored during the search.

LaTeX output for Clifford units looks like `\clifford[1]{e}^{{\nu}}`, where `1` is the `representation_label` and `\nu` is the index of the corresponding unit. This provides a flexible typesetting with a suitable definition of the `\clifford` command. For example, the definition

```
\newcommand{\clifford}[1][]{}
```

typesets all Clifford units identically, while the alternative definition

```
\newcommand{\clifford}[2][]{\ifcase #1 #2\or \tilde{#2} \or \breve{#2} \fi}
```

prints units with `representation_label=0` as e, with `representation_label=1` as $\tilde{e}$ and with `representation_label=2` as $\check{e}$.

4.15.2 Color algebra

For computations in quantum chromodynamics, GiNaC implements the base elements and structure constants of the su(3) Lie algebra (color algebra). The base elements T_a are constructed by the function

```
ex color_T(const ex & a, unsigned char rl = 0);
```

which takes two arguments: the index and a *representation label* in the range 0 to 255 which is used to distinguish elements of different color algebras. Objects with different labels commutate with each other. The dimension of the index must be exactly 8 and it should be of class `idx`, not `varidx`.

The unity element of a color algebra is constructed by

```
ex color_ONE(unsigned char rl = 0);
```

Please notice: You must always use `color_ONE()` when referring to multiples of the unity element, even though it's customary to omit it. E.g. instead of `color_T(a)*(color_T(b)*indexed(X,b)+1)` you have to write `color_T(a)*(color_T(b)*indexed(X,b)+color_ONE())`. Otherwise, GiNaC may produce incorrect results.

The functions

```
ex color_d(const ex & a, const ex & b, const ex & c);
ex color_f(const ex & a, const ex & b, const ex & c);
```

create the symmetric and antisymmetric structure constants d_abc and f_abc which satisfy $\{T_a, T_b\} = 1/3 delta_a b + d_abcT_c$ and $[T_a, T_b] = if_abcT_c$.

These functions evaluate to their numerical values, if you supply numeric indices to them. The index values should be in the range from 1 to 8, not from 0 to 7. This departure from usual conventions goes along better with the notations used in physical literature.

There's an additional function

```
ex color_h(const ex & a, const ex & b, const ex & c);
```

which returns the linear combination '`color_d(a, b, c)+I*color_f(a, b, c)`'.

The function `simplify_indexed()` performs some simplifications on expressions containing color objects:

```
{
    ...
    idx a(symbol("a"), 8), b(symbol("b"), 8), c(symbol("c"), 8),
        k(symbol("k"), 8), l(symbol("l"), 8);

    e = color_d(a, b, l) * color_f(a, b, k);
    cout << e.simplify_indexed() << endl;
     // -> 0

    e = color_d(a, b, l) * color_d(a, b, k);
    cout << e.simplify_indexed() << endl;
     // -> 5/3*delta.k.l

    e = color_f(l, a, b) * color_f(a, b, k);
    cout << e.simplify_indexed() << endl;
     // -> 3*delta.k.l

    e = color_h(a, b, c) * color_h(a, b, c);
```

```
cout << e.simplify_indexed() << endl;
 // -> -32/3

e = color_h(a, b, c) * color_T(b) * color_T(c);
cout << e.simplify_indexed() << endl;
 // -> -2/3*T.a

e = color_h(a, b, c) * color_T(a) * color_T(b) * color_T(c);
cout << e.simplify_indexed() << endl;
 // -> -8/9*ONE

e = color_T(k) * color_T(a) * color_T(b) * color_T(k);
cout << e.simplify_indexed() << endl;
 // -> 1/4*delta.b.a*ONE-1/6*T.a*T.b
...
```

To calculate the trace of an expression containing color objects you use one of the functions

```
ex color_trace(const ex & e, const std::set<unsigned char> & rls);
ex color_trace(const ex & e, const lst & rll);
ex color_trace(const ex & e, unsigned char rl = 0);
```

These functions take the trace over all color '`T`' objects in the specified set `rls` or list `rll` of representation labels, or the single label `rl`; '`T`'s with other labels are left standing. For example:

```
...
e = color_trace(4 * color_T(a) * color_T(b) * color_T(c));
cout << e << endl;
 // -> -I*f.a.c.b+d.a.c.b
}
```

5 Methods and functions

In this chapter the most important algorithms provided by GiNaC will be described. Some of them are implemented as functions on expressions, others are implemented as methods provided by expression objects. If they are methods, there exists a wrapper function around it, so you can alternatively call it in a functional way as shown in the simple example:

```
...
cout << "As method:   " << sin(1).evalf() << endl;
cout << "As function: " << evalf(sin(1)) << endl;
...
```

The general rule is that wherever methods accept one or more parameters (*arg1*, *arg2*, ...) the order of arguments the function wrapper accepts is the same but preceded by the object to act on (*object*, *arg1*, *arg2*, ...). This approach is the most natural one in an OO model but it may lead to confusion for MapleV users because where they would type `A:=x+1; subs(x=2,A);` GiNaC would require `A=x+1; subs(A,x==2);` (after proper declaration of `A` and `x`). On the other hand, since MapleV returns 3 on `A:=x^2+3; coeff(A,x,0);` (GiNaC: `A=pow(x,2)+3; coeff(A,x,0);`) it is clear that MapleV is not trying to be consistent here. Also, users of MuPAD will in most cases feel more comfortable with GiNaC's convention. All function wrappers are implemented as simple inline functions which just call the corresponding method and are only provided for users uncomfortable with OO who are dead set to avoid method invocations. Generally, nested function wrappers are much harder to read than a sequence of methods and should therefore be avoided if possible. On the other hand, not everything in GiNaC is a method on class `ex` and sometimes calling a function cannot be avoided.

5.1 Getting information about expressions

5.1.1 Checking expression types

Sometimes it's useful to check whether a given expression is a plain number, a sum, a polynomial with integer coefficients, or of some other specific type. GiNaC provides a couple of functions for this:

```
bool is_a<T>(const ex & e);
bool is_exactly_a<T>(const ex & e);
bool ex::info(unsigned flag);
unsigned ex::return_type() const;
return_type_t ex::return_type_tinfo() const;
```

When the test made by `is_a<T>()` returns true, it is safe to call one of the functions `ex_to<T>()`, where `T` is one of the class names (See Section 4.4 [The class hierarchy], page 12, for a list of all classes). For example, assuming `e` is an `ex`:

```
{
    ...
    if (is_a<numeric>(e))
        numeric n = ex_to<numeric>(e);
    ...
}
```

`is_a<T>(e)` allows you to check whether the top-level object of an expression '`e`' is an instance of the GiNaC class '`T`' (See Section 4.4 [The class hierarchy], page 12, for a list of all classes). This is most useful, e.g., for checking whether an expression is a number, a sum, or a product:

```
{
    symbol x("x");
    ex e1 = 42;
```

```
        ex e2 = 4*x - 3;
        is_a<numeric>(e1);  // true
        is_a<numeric>(e2);  // false
        is_a<add>(e1);      // false
        is_a<add>(e2);      // true
        is_a<mul>(e1);      // false
        is_a<mul>(e2);      // false
    }
```

In contrast, `is_exactly_a<T>(e)` allows you to check whether the top-level object of an expression '`e`' is an instance of the GiNaC class '`T`', not including parent classes.

The `info()` method is used for checking certain attributes of expressions. The possible values for the `flag` argument are defined in `ginac/flags.h`, the most important being explained in the following table:

Flag	**Returns true if the object is...**
`numeric`	...a number (same as `is_a<numeric>(...)`)
`real`	...a real number, symbol or constant (i.e. is not complex)
`rational`	...an exact rational number (integers are rational, too)
`integer`	...a (non-complex) integer
`crational`	...an exact (complex) rational number (such as $2/3 + 7/2 * I$)
`cinteger`	...a (complex) integer (such as $2 - 3 * I$)
`positive`	...not complex and greater than 0
`negative`	...not complex and less than 0
`nonnegative`	...not complex and greater than or equal to 0
`posint`	...an integer greater than 0
`negint`	...an integer less than 0
`nonnegint`	...an integer greater than or equal to 0
`even`	...an even integer
`odd`	...an odd integer
`prime`	...a prime integer (probabilistic primality test)
`relation`	...a relation (same as `is_a<relational>(...)`)
`relation_equal`	...a `==` relation
`relation_not_equal`	...a `!=` relation
`relation_less`	...a `<` relation
`relation_less_or_equal`	...a `<=` relation
`relation_greater`	...a `>` relation
`relation_greater_or_equal`	...a `>=` relation
`symbol`	...a symbol (same as `is_a<symbol>(...)`)
`list`	...a list (same as `is_a<lst>(...)`)
`polynomial`	...a polynomial (i.e. only consists of sums and products of numbers and symbols with positive integer powers)
`integer_polynomial`	...a polynomial with (non-complex) integer coefficients
`cinteger_polynomial`	...a polynomial with (possibly complex) integer coefficients (such as $2 - 3 * I$)
`rational_polynomial`	...a polynomial with (non-complex) rational coefficients
`crational_polynomial`	...a polynomial with (possibly complex) rational coefficients (such as $2/3 + 7/2 * I$)
`rational_function`	...a rational function ($x + y$, $z/(x + y)$)

To determine whether an expression is commutative or non-commutative and if so, with which other expressions it would commutate, you use the methods `return_type()` and `return_type_tinfo()`. See Section 4.15 [Non-commutative objects], page 39, for an explanation of these.

5.1.2 Accessing subexpressions

Many GiNaC classes, like `add`, `mul`, `lst`, and `function`, act as containers for subexpressions. For example, the subexpressions of a sum (an `add` object) are the individual terms, and the subexpressions of a `function` are the function's arguments.

GiNaC provides several ways of accessing subexpressions. The first way is to use the two methods

```
size_t ex::nops();
ex ex::op(size_t i);
```

`nops()` determines the number of subexpressions (operands) contained in the expression, while `op(i)` returns the `i`-th (0..`nops()-1`) subexpression. In the case of a `power` object, `op(0)` will return the basis and `op(1)` the exponent. For `indexed` objects, `op(0)` is the base expression and `op(i)`, $i > 0$ are the indices.

The second way to access subexpressions is via the STL-style random-access iterator class `const_iterator` and the methods

```
const_iterator ex::begin();
const_iterator ex::end();
```

`begin()` returns an iterator referring to the first subexpression; `end()` returns an iterator which is one-past the last subexpression. If the expression has no subexpressions, then `begin() == end()`. These iterators can also be used in conjunction with non-modifying STL algorithms.

Here is an example that (non-recursively) prints the subexpressions of a given expression in three different ways:

```
{
    ex e = ...

    // with nops()/op()
    for (size_t i = 0; i != e.nops(); ++i)
        cout << e.op(i) << endl;

    // with iterators
    for (const_iterator i = e.begin(); i != e.end(); ++i)
        cout << *i << endl;

    // with iterators and STL copy()
    std::copy(e.begin(), e.end(), std::ostream_iterator<ex>(cout, "\n"));
}
```

`op()`/`nops()` and `const_iterator` only access an expression's immediate children. GiNaC provides two additional iterator classes, `const_preorder_iterator` and `const_postorder_iterator`, that iterate over all objects in an expression tree, in preorder or postorder, respectively. They are STL-style forward iterators, and are created with the methods

```
const_preorder_iterator ex::preorder_begin();
const_preorder_iterator ex::preorder_end();
const_postorder_iterator ex::postorder_begin();
const_postorder_iterator ex::postorder_end();
```

The following example illustrates the differences between `const_iterator`, `const_preorder_iterator`, and `const_postorder_iterator`:

```
{
```

```
    symbol A("A"), B("B"), C("C");
    ex e = lst{lst{A, B}, C};

    std::copy(e.begin(), e.end(),
              std::ostream_iterator<ex>(cout, "\n"));
    // {A,B}
    // C

    std::copy(e.preorder_begin(), e.preorder_end(),
              std::ostream_iterator<ex>(cout, "\n"));
    // {{A,B},C}
    // {A,B}
    // A
    // B
    // C

    std::copy(e.postorder_begin(), e.postorder_end(),
              std::ostream_iterator<ex>(cout, "\n"));
    // A
    // B
    // {A,B}
    // C
    // {{A,B},C}
}
```

Finally, the left-hand side and right-hand side expressions of objects of class `relational` (and only of these) can also be accessed with the methods

```
ex ex::lhs();
ex ex::rhs();
```

5.1.3 Comparing expressions

Expressions can be compared with the usual C++ relational operators like `==`, `>`, and `<` but if the expressions contain symbols, the result is usually not determinable and the result will be `false`, except in the case of the `!=` operator. You should also be aware that GiNaC will only do the most trivial test for equality (subtracting both expressions), so something like `(pow(x,2)+x)/x==x+1` will return `false`.

Actually, if you construct an expression like `a == b`, this will be represented by an object of the `relational` class (see Section 4.11 [Relations], page 24) which is not evaluated until (explicitly or implicitly) cast to a `bool`.

There are also two methods

```
bool ex::is_equal(const ex & other);
bool ex::is_zero();
```

for checking whether one expression is equal to another, or equal to zero, respectively. See also the method `ex::is_zero_matrix()`, see Section 4.13 [Matrices], page 26.

5.1.4 Ordering expressions

Sometimes it is necessary to establish a mathematically well-defined ordering on a set of arbitrary expressions, for example to use expressions as keys in a `std::map<>` container, or to bring a vector of expressions into a canonical order (which is done internally by GiNaC for sums and products).

The operators `<`, `>` etc. described in the last section cannot be used for this, as they don't implement an ordering relation in the mathematical sense. In particular, they are not guaranteed to be antisymmetric: if 'a' and 'b' are different expressions, and `a < b` yields `false`, then `b < a` doesn't necessarily yield `true`.

By default, STL classes and algorithms use the `<` and `==` operators to compare objects, which are unsuitable for expressions, but GiNaC provides two functors that can be supplied as proper binary comparison predicates to the STL:

```
class ex_is_less {
public:
    bool operator()(const ex &lh, const ex &rh) const;
};

class ex_is_equal {
public:
    bool operator()(const ex &lh, const ex &rh) const;
};
```

For example, to define a `map` that maps expressions to strings you have to use

```
std::map<ex, std::string, ex_is_less> myMap;
```

Omitting the `ex_is_less` template parameter will introduce spurious bugs because the map operates improperly.

Other examples for the use of the functors:

```
std::vector<ex> v;
// fill vector
...

// sort vector
std::sort(v.begin(), v.end(), ex_is_less());

// count the number of expressions equal to '1'
unsigned num_ones = std::count_if(v.begin(), v.end(),
                                  [](const ex& e) { return ex_is_equal()(e, 1); });
```

The implementation of `ex_is_less` uses the member function

```
int ex::compare(const ex & other) const;
```

which returns 0 if `*this` and `other` are equal, -1 if `*this` sorts before `other`, and 1 if `*this` sorts after `other`.

5.2 Numerical evaluation

GiNaC keeps algebraic expressions, numbers and constants in their exact form. To evaluate them using floating-point arithmetic you need to call

```
ex ex::evalf() const;
```

The accuracy of the evaluation is controlled by the global object `Digits` which can be assigned an integer value. The default value of `Digits` is 17. See Section 4.6 [Numbers], page 16, for more information and examples.

To evaluate an expression to a `double` floating-point number you can call `evalf()` followed by `numeric::to_double()`, like this:

```
{
    // Approximate sin(x/Pi)
    symbol x("x");
```

```
    ex e = series(sin(x/Pi), x == 0, 6);

    // Evaluate numerically at x=0.1
    ex f = evalf(e.subs(x == 0.1));

    // ex_to<numeric> is an unsafe cast, so check the type first
    if (is_a<numeric>(f)) {
        double d = ex_to<numeric>(f).to_double();
        cout << d << endl;
         // -> 0.0318256
    } else
        // error
}
```

5.3 Substituting expressions

Algebraic objects inside expressions can be replaced with arbitrary expressions via the `.subs()` method:

```
ex ex::subs(const ex & e, unsigned options = 0);
ex ex::subs(const exmap & m, unsigned options = 0);
ex ex::subs(const lst & syms, const lst & repls, unsigned options = 0);
```

In the first form, `subs()` accepts a relational of the form '`object == expression`' or a `lst` of such relationals:

```
{
    symbol x("x"), y("y");

    ex e1 = 2*x*x-4*x+3;
    cout << "e1(7) = " << e1.subs(x == 7) << endl;
     // -> 73

    ex e2 = x*y + x;
    cout << "e2(-2, 4) = " << e2.subs(lst{x == -2, y == 4}) << endl;
     // -> -10
}
```

If you specify multiple substitutions, they are performed in parallel, so e.g. `subs(lst{x == y, y == x})` exchanges '`x`' and '`y`'.

The second form of `subs()` takes an `exmap` object which is a pair associative container that maps expressions to expressions (currently implemented as a `std::map`). This is the most efficient one of the three `subs()` forms and should be used when the number of objects to be substituted is large or unknown.

Using this form, the second example from above would look like this:

```
{
    symbol x("x"), y("y");
    ex e2 = x*y + x;

    exmap m;
    m[x] = -2;
    m[y] = 4;
    cout << "e2(-2, 4) = " << e2.subs(m) << endl;
}
```

The third form of `subs()` takes two lists, one for the objects to be replaced and one for the expressions to be substituted (both lists must contain the same number of elements). Using this form, you would write

```
{
    symbol x("x"), y("y");
    ex e2 = x*y + x;

    cout << "e2(-2, 4) = " << e2.subs(lst{x, y}, lst{-2, 4}) << endl;
}
```

The optional last argument to `subs()` is a combination of `subs_options` flags. There are three options available: `subs_options::no_pattern` disables pattern matching, which makes large `subs()` operations significantly faster if you are not using patterns. The second option, `subs_options::algebraic` enables algebraic substitutions in products and powers. See Section 5.4 [Pattern matching and advanced substitutions], page 54, for more information about patterns and algebraic substitutions. The third option, `subs_options::no_index_renaming` disables the feature that dummy indices are renamed if the substitution could give a result in which a dummy index occurs more than two times. This is sometimes necessary if you want to use `subs()` to rename your dummy indices.

`subs()` performs syntactic substitution of any complete algebraic object; it does not try to match sub-expressions as is demonstrated by the following example:

```
{
    symbol x("x"), y("y"), z("z");

    ex e1 = pow(x+y, 2);
    cout << e1.subs(x+y == 4) << endl;
     // -> 16

    ex e2 = sin(x)*sin(y)*cos(x);
    cout << e2.subs(sin(x) == cos(x)) << endl;
     // -> cos(x)^2*sin(y)

    ex e3 = x+y+z;
    cout << e3.subs(x+y == 4) << endl;
     // -> x+y+z
     // (and not 4+z as one might expect)
}
```

A more powerful form of substitution using wildcards is described in the next section.

5.4 Pattern matching and advanced substitutions

GiNaC allows the use of patterns for checking whether an expression is of a certain form or contains subexpressions of a certain form, and for substituting expressions in a more general way.

A *pattern* is an algebraic expression that optionally contains wildcards. A *wildcard* is a special kind of object (of class `wildcard`) that represents an arbitrary expression. Every wildcard has a *label* which is an unsigned integer number to allow having multiple different wildcards in a pattern. Wildcards are printed as '`$label`' (this is also the way they are specified in `ginsh`). In C++ code, wildcard objects are created with the call

```
ex wild(unsigned label = 0);
```

which is simply a wrapper for the `wildcard()` constructor with a shorter name.

Some examples for patterns:

Constructed as	**Output as**
`wild()`	'`$0`'
`pow(x,wild())`	'`x^$0`'
`atan2(wild(1),wild(2))`	'`atan2($1,$2)`'
`indexed(A,idx(wild(),3))`	'`A.$0`'

Notes:

- Wildcards behave like symbols and are subject to the same algebraic rules. E.g., '`$0+2*$0`' is automatically transformed to '`3*$0`'.
- As shown in the last example, to use wildcards for indices you have to use them as the value of an `idx` object. This is because indices must always be of class `idx` (or a subclass).
- Wildcards only represent expressions or subexpressions. It is not possible to use them as placeholders for other properties like index dimension or variance, representation labels, symmetry of indexed objects etc.
- Because wildcards are commutative, it is not possible to use wildcards as part of noncommutative products.
- A pattern does not have to contain wildcards. '`x`' and '`x+y`' are also valid patterns.

5.4.1 Matching expressions

The most basic application of patterns is to check whether an expression matches a given pattern. This is done by the function

```
bool ex::match(const ex & pattern);
bool ex::match(const ex & pattern, exmap& repls);
```

This function returns `true` when the expression matches the pattern and `false` if it doesn't. If used in the second form, the actual subexpressions matched by the wildcards get returned in the associative array `repls` with '`wildcard`' as a key. If `match()` returns false, `repls` remains unmodified.

The matching algorithm works as follows:

- A single wildcard matches any expression. If one wildcard appears multiple times in a pattern, it must match the same expression in all places (e.g. '`$0`' matches anything, and '`$0*($0+1)`' matches '`x*(x+1)`' but not '`x*(y+1)`').
- If the expression is not of the same class as the pattern, the match fails (i.e. a sum only matches a sum, a function only matches a function, etc.).
- If the pattern is a function, it only matches the same function (i.e. '`sin($0)`' matches '`sin(x)`' but doesn't match '`exp(x)`').
- Except for sums and products, the match fails if the number of subexpressions (`nops()`) is not equal to the number of subexpressions of the pattern.
- If there are no subexpressions, the expressions and the pattern must be equal (in the sense of `is_equal()`).
- Except for sums and products, each subexpression (`op()`) must match the corresponding subexpression of the pattern.

Sums (`add`) and products (`mul`) are treated in a special way to account for their commutativity and associativity:

- If the pattern contains a term or factor that is a single wildcard, this one is used as the *global wildcard*. If there is more than one such wildcard, one of them is chosen as the global wildcard in a random way.

- Every term/factor of the pattern, except the global wildcard, is matched against every term of the expression in sequence. If no match is found, the whole match fails. Terms that did match are not considered in further matches.
- If there are no unmatched terms left, the match succeeds. Otherwise the match fails unless there is a global wildcard in the pattern, in which case this wildcard matches the remaining terms.

In general, having more than one single wildcard as a term of a sum or a factor of a product (such as '`a+$0+$1`') will lead to unpredictable or ambiguous results.

Here are some examples in `ginsh` to demonstrate how it works (the `match()` function in `ginsh` returns '`FAIL`' if the match fails, and the list of wildcard replacements otherwise):

```
> match((x+y)^a,(x+y)^a);
{}
> match((x+y)^a,(x+y)^b);
FAIL
> match((x+y)^a,$1^$2);
{$1==x+y,$2==a}
> match((x+y)^a,$1^$1);
FAIL
> match((x+y)^(x+y),$1^$1);
{$1==x+y}
> match((x+y)^(x+y),$1^$2);
{$1==x+y,$2==x+y}
> match((a+b)*(a+c),($1+b)*($1+c));
{$1==a}
> match((a+b)*(a+c),(a+$1)*(a+$2));
{$1==b,$2==c}
  (Unpredictable. The result might also be [$1==c,$2==b].)
> match((a+b)*(a+c),($1+$2)*($1+$3));
  (The result is undefined. Due to the sequential nature of the algorithm
   and the re-ordering of terms in GiNaC, the match for the first factor
   may be {$1==a,$2==b} in which case the match for the second factor
   succeeds, or it may be {$1==b,$2==a} which causes the second match to
   fail.)
> match(a*(x+y)+a*z+b,a*$1+$2);
  (This is also ambiguous and may return either {$1==z,$2==a*(x+y)+b} or
   {$1=x+y,$2=a*z+b}.)
> match(a+b+c+d+e+f,c);
FAIL
> match(a+b+c+d+e+f,c+$0);
{$0==a+e+b+f+d}
> match(a+b+c+d+e+f,c+e+$0);
{$0==a+b+f+d}
> match(a+b,a+b+$0);
{$0==0}
> match(a*b^2,a^$1*b^$2);
FAIL
  (The matching is syntactic, not algebraic, and "a" doesn't match "a^$1"
   even though a==a^1.)
> match(x*atan2(x,x^2),$0*atan2($0,$0^2));
{$0==x}
> match(atan2(y,x^2),atan2(y,$0));
```

```
{$0==x^2}
```

5.4.2 Matching parts of expressions

A more general way to look for patterns in expressions is provided by the member function

```
bool ex::has(const ex & pattern);
```

This function checks whether a pattern is matched by an expression itself or by any of its subexpressions.

Again some examples in `ginsh` for illustration (in `ginsh`, `has()` returns '`1`' for `true` and '`0`' for `false`):

```
> has(x*sin(x+y+2*a),y);
1
> has(x*sin(x+y+2*a),x+y);
0
  (This is because in GiNaC, "x+y" is not a subexpression of "x+y+2*a" (which
   has the subexpressions "x", "y" and "2*a".)
> has(x*sin(x+y+2*a),x+y+$1);
1
  (But this is possible.)
> has(x*sin(2*(x+y)+2*a),x+y);
0
  (This fails because "2*(x+y)" automatically gets converted to "2*x+2*y" of
   which "x+y" is not a subexpression.)
> has(x+1,x^$1);
0
  (Although x^1==x and x^0==1, neither "x" nor "1" are actually of the form
   "x^something".)
> has(4*x^2-x+3,$1*x);
1
> has(4*x^2+x+3,$1*x);
0
  (Another possible pitfall. The first expression matches because the term
   "-x" has the form "(-1)*x" in GiNaC. To check whether a polynomial
   contains a linear term you should use the coeff() function instead.)
```

The method

```
bool ex::find(const ex & pattern, exset& found);
```

works a bit like `has()` but it doesn't stop upon finding the first match. Instead, it appends all found matches to the specified list. If there are multiple occurrences of the same expression, it is entered only once to the list. `find()` returns false if no matches were found (in `ginsh`, it returns an empty list):

```
> find(1+x+x^2+x^3,x);
{x}
> find(1+x+x^2+x^3,y);
{}
> find(1+x+x^2+x^3,x^$1);
{x^3,x^2}
  (Note the absence of "x".)
> expand((sin(x)+sin(y))*(a+b));
sin(y)*a+sin(x)*b+sin(x)*a+sin(y)*b
> find(%,sin($1));
{sin(y),sin(x)}
```

5.4.3 Substituting expressions

Probably the most useful application of patterns is to use them for substituting expressions with the `subs()` method. Wildcards can be used in the search patterns as well as in the replacement expressions, where they get replaced by the expressions matched by them. `subs()` doesn't know anything about algebra; it performs purely syntactic substitutions.

Some examples:

```
> subs(a^2+b^2+(x+y)^2,$1^2==$1^3);
b^3+a^3+(x+y)^3
> subs(a^4+b^4+(x+y)^4,$1^2==$1^3);
b^4+a^4+(x+y)^4
> subs((a+b+c)^2,a+b==x);
(a+b+c)^2
> subs((a+b+c)^2,a+b+$1==x+$1);
(x+c)^2
> subs(a+2*b,a+b==x);
a+2*b
> subs(4*x^3-2*x^2+5*x-1,x==a);
-1+5*a-2*a^2+4*a^3
> subs(4*x^3-2*x^2+5*x-1,x^$0==a^$0);
-1+5*x-2*a^2+4*a^3
> subs(sin(1+sin(x)),sin($1)==cos($1));
cos(1+cos(x))
> expand(subs(a*sin(x+y)^2+a*cos(x+y)^2+b,cos($1)^2==1-sin($1)^2));
a+b
```

The last example would be written in C++ in this way:

```
{
    symbol a("a"), b("b"), x("x"), y("y");
    e = a*pow(sin(x+y), 2) + a*pow(cos(x+y), 2) + b;
    e = e.subs(pow(cos(wild()), 2) == 1-pow(sin(wild()), 2));
    cout << e.expand() << endl;
     // -> a+b
}
```

5.4.4 The option algebraic

Both `has()` and `subs()` take an optional argument to pass them extra options. This section describes what happens if you give the former the option `has_options::algebraic` or the latter `subs_options::algebraic`. In that case the matching condition for powers and multiplications is changed in such a way that they become more intuitive. Intuition says that `x*y` is a part of `x*y*z`. If you use these options you will find that `(x*y*z).has(x*y, has_options::algebraic)` indeed returns true. Besides matching some of the factors of a product also powers match as often as is possible without getting negative exponents. For example `(x^5*y^2*z).subs(x^2*y^2==c, subs_options::algebraic)` will return `x*c^2*z`. This also works with negative powers: `(x^(-3)*y^(-2)*z).subs(1/(x*y)==c, subs_options::algebraic)` will return `x^(-1)*c^2*z`.

Please notice: this only works for multiplications and not for locating `x+y` within `x+y+z`.

5.5 Applying a function on subexpressions

Sometimes you may want to perform an operation on specific parts of an expression while leaving the general structure of it intact. An example of this would be a matrix trace operation: the trace of a sum is the sum of the traces of the individual terms. That is, the trace should *map*

on the sum, by applying itself to each of the sum's operands. It is possible to do this manually which usually results in code like this:

```
ex calc_trace(ex e)
{
    if (is_a<matrix>(e))
        return ex_to<matrix>(e).trace();
    else if (is_a<add>(e)) {
        ex sum = 0;
        for (size_t i=0; i<e.nops(); i++)
            sum += calc_trace(e.op(i));
        return sum;
    } else if (is_a<mul>)(e)) {
        ...
    } else {
        ...
    }
}
```

This is, however, slightly inefficient (if the sum is very large it can take a long time to add the terms one-by-one), and its applicability is limited to a rather small class of expressions. If `calc_trace()` is called with a relation or a list as its argument, you will probably want the trace to be taken on both sides of the relation or of all elements of the list.

GiNaC offers the `map()` method to aid in the implementation of such operations:

```
ex ex::map(map_function & f) const;
ex ex::map(ex (*f)(const ex & e)) const;
```

In the first (preferred) form, `map()` takes a function object that is subclassed from the `map_function` class. In the second form, it takes a pointer to a function that accepts and returns an expression. `map()` constructs a new expression of the same type, applying the specified function on all subexpressions (in the sense of `op()`), non-recursively.

The use of a function object makes it possible to supply more arguments to the function that is being mapped, or to keep local state information. The `map_function` class declares a virtual function call operator that you can overload. Here is a sample implementation of `calc_trace()` that uses `map()` in a recursive fashion:

```
struct calc_trace : public map_function {
    ex operator()(const ex &e)
    {
        if (is_a<matrix>(e))
            return ex_to<matrix>(e).trace();
        else if (is_a<mul>(e)) {
            ...
        } else
            return e.map(*this);
    }
};
```

This function object could then be used like this:

```
{
    ex M = ... // expression with matrices
    calc_trace do_trace;
    ex tr = do_trace(M);
}
```

Here is another example for you to meditate over. It removes quadratic terms in a variable from an expanded polynomial:

```
struct map_rem_quad : public map_function {
    ex var;
    map_rem_quad(const ex & var_) : var(var_) {}

    ex operator()(const ex & e)
    {
        if (is_a<add>(e) || is_a<mul>(e))
          return e.map(*this);
        else if (is_a<power>(e) &&
                 e.op(0).is_equal(var) && e.op(1).info(info_flags::even))
            return 0;
        else
            return e;
    }
};

...

{
    symbol x("x"), y("y");

    ex e;
    for (int i=0; i<8; i++)
        e += pow(x, i) * pow(y, 8-i) * (i+1);
    cout << e << endl;
     // -> 4*y^5*x^3+5*y^4*x^4+8*y*x^7+7*y^2*x^6+2*y^7*x+6*y^3*x^5+3*y^6*x^2+y^8

    map_rem_quad rem_quad(x);
    cout << rem_quad(e) << endl;
     // -> 4*y^5*x^3+8*y*x^7+2*y^7*x+6*y^3*x^5+y^8
}
```

`ginsh` offers a slightly different implementation of `map()` that allows applying algebraic functions to operands. The second argument to `map()` is an expression containing the wildcard '`$0`' which acts as the placeholder for the operands:

```
> map(a*b,sin($0));
sin(a)*sin(b)
> map(a+2*b,sin($0));
sin(a)+sin(2*b)
> map({a,b,c},$0^2+$0);
{a^2+a,b^2+b,c^2+c}
```

Note that it is only possible to use algebraic functions in the second argument. You can not use functions like '`diff()`', '`op()`', '`subs()`' etc. because these are evaluated immediately:

```
> map({a,b,c},diff($0,a));
{0,0,0}
  This is because "diff($0,a)" evaluates to "0", so the command is equivalent
  to "map({a,b,c},0)".
```

5.6 Visitors and tree traversal

Suppose that you need a function that returns a list of all indices appearing in an arbitrary expression. The indices can have any dimension, and for indices with variance you always want the covariant version returned.

You can't use `get_free_indices()` because you also want to include dummy indices in the list, and you can't use `find()` as it needs specific index dimensions (and it would require two passes: one for indices with variance, one for plain ones).

The obvious solution to this problem is a tree traversal with a type switch, such as the following:

```
void gather_indices_helper(const ex & e, lst & l)
{
    if (is_a<varidx>(e)) {
        const varidx & vi = ex_to<varidx>(e);
        l.append(vi.is_covariant() ? vi : vi.toggle_variance());
    } else if (is_a<idx>(e)) {
        l.append(e);
    } else {
        size_t n = e.nops();
        for (size_t i = 0; i < n; ++i)
            gather_indices_helper(e.op(i), l);
    }
}

lst gather_indices(const ex & e)
{
    lst l;
    gather_indices_helper(e, l);
    l.sort();
    l.unique();
    return l;
}
```

This works fine but fans of object-oriented programming will feel uncomfortable with the type switch. One reason is that there is a possibility for subtle bugs regarding derived classes. If we had, for example, written

```
if (is_a<idx>(e)) {
  ...
} else if (is_a<varidx>(e)) {
  ...
```

in `gather_indices_helper`, the code wouldn't have worked because the first line "absorbs" all classes derived from `idx`, including `varidx`, so the special case for `varidx` would never have been executed.

Also, for a large number of classes, a type switch like the above can get unwieldy and inefficient (it's a linear search, after all). `gather_indices_helper` only checks for two classes, but if you had to write a function that required a different implementation for nearly every GiNaC class, the result would be very hard to maintain and extend.

The cleanest approach to the problem would be to add a new virtual function to GiNaC's class hierarchy. In our example, there would be specializations for `idx` and `varidx` while the default implementation in `basic` performed the tree traversal. Unfortunately, in C++ it's impossible to add virtual member functions to existing classes without changing their source and recompiling everything. GiNaC comes with source, so you could actually do this, but for a small algorithm like the one presented this would be impractical.

One solution to this dilemma is the *Visitor* design pattern, which is implemented in GiNaC (actually, Robert Martin's Acyclic Visitor variation, described in detail in `https://condor.depaul.edu/dmumaugh/OOT/Design-Principles/acv.pdf`). Instead of adding virtual functions to the class hierarchy to implement operations, GiNaC provides a single "bouncing" method `accept()` that takes an instance of a special `visitor` class and redirects execution to the one `visit()` virtual function of the visitor that matches the type of object that `accept()` was being invoked on.

Visitors in GiNaC must derive from the global `visitor` class as well as from the class `T::visitor` of each class `T` they want to visit, and implement the member functions `void visit(const T &)` for each class.

A call of

```
void ex::accept(visitor & v) const;
```

will then dispatch to the correct `visit()` member function of the specified visitor `v` for the type of GiNaC object at the root of the expression tree (e.g. a `symbol`, an `idx` or a `mul`).

Here is an example of a visitor:

```
class my_visitor
 : public visitor,          // this is required
   public add::visitor,     // visit add objects
   public numeric::visitor, // visit numeric objects
   public basic::visitor    // visit basic objects
{
    void visit(const add & x)
    { cout << "called with an add object" << endl; }

    void visit(const numeric & x)
    { cout << "called with a numeric object" << endl; }

    void visit(const basic & x)
    { cout << "called with a basic object" << endl; }
};
```

which can be used as follows:

```
...
    symbol x("x");
    ex e1 = 42;
    ex e2 = 4*x-3;
    ex e3 = 8*x;

    my_visitor v;
    e1.accept(v);
     // prints "called with a numeric object"
    e2.accept(v);
     // prints "called with an add object"
    e3.accept(v);
     // prints "called with a basic object"
...
```

The `visit(const basic &)` method gets called for all objects that are not `numeric` or `add` and acts as an (optional) default.

From a conceptual point of view, the `visit()` methods of the visitor behave like a newly added virtual function of the visited hierarchy. In addition, visitors can store state in member variables,

and they can be extended by deriving a new visitor from an existing one, thus building hierarchies of visitors.

We can now rewrite our index example from above with a visitor:

```
class gather_indices_visitor
 : public visitor, public idx::visitor, public varidx::visitor
{
    lst l;

    void visit(const idx & i)
    {
        l.append(i);
    }

    void visit(const varidx & vi)
    {
        l.append(vi.is_covariant() ? vi : vi.toggle_variance());
    }

public:
    const lst & get_result() // utility function
    {
        l.sort();
        l.unique();
        return l;
    }
};
```

What's missing is the tree traversal. We could implement it in `visit(const basic &)`, but GiNaC has predefined methods for this:

```
void ex::traverse_preorder(visitor & v) const;
void ex::traverse_postorder(visitor & v) const;
void ex::traverse(visitor & v) const;
```

`traverse_preorder()` visits a node *before* visiting its subexpressions, while `traverse_postorder()` visits a node *after* visiting its subexpressions. `traverse()` is a synonym for `traverse_preorder()`.

Here is a new implementation of `gather_indices()` that uses the visitor and `traverse()`:

```
lst gather_indices(const ex & e)
{
    gather_indices_visitor v;
    e.traverse(v);
    return v.get_result();
}
```

Alternatively, you could use pre- or postorder iterators for the tree traversal:

```
lst gather_indices(const ex & e)
{
    gather_indices_visitor v;
    for (const_preorder_iterator i = e.preorder_begin();
         i != e.preorder_end(); ++i) {
        i->accept(v);
    }
    return v.get_result();
```

```
}
```

5.7 Polynomial arithmetic

5.7.1 Testing whether an expression is a polynomial

Testing whether an expression is a polynomial in one or more variables can be done with the method

```
bool ex::is_polynomial(const ex & vars) const;
```

In the case of more than one variable, the variables are given as a list.

```
(x*y*sin(y)).is_polynomial(x)          // Returns true.
(x*y*sin(y)).is_polynomial(lst{x,y})   // Returns false.
```

5.7.2 Expanding and collecting

A polynomial in one or more variables has many equivalent representations. Some useful ones serve a specific purpose. Consider for example the trivariate polynomial $4*x*y+x*z+20*y^2+21*y*z+4*z^2$ (written down here in output-style). It is equivalent to the factorized polynomial $(x+5*y+4*z)*(4*y+z)$. Other representations are the recursive ones where one collects for exponents in one of the three variable. Since the factors are themselves polynomials in the remaining two variables the procedure can be repeated. In our example, two possibilities would be $(4*y+z)*x+20*y^2+21*y*z+4*z^2$ and $20*y^2+(21*z+4*x)*y+4*z^2+x*z$.

To bring an expression into expanded form, its method

```
ex ex::expand(unsigned options = 0);
```

may be called. In our example above, this corresponds to $4*x*y+x*z+20*y^2+21*y*z+4*z^2$. Again, since the canonical form in GiNaC is not easy to guess you should be prepared to see different orderings of terms in such sums!

Another useful representation of multivariate polynomials is as a univariate polynomial in one of the variables with the coefficients being polynomials in the remaining variables. The method `collect()` accomplishes this task:

```
ex ex::collect(const ex & s, bool distributed = false);
```

The first argument to `collect()` can also be a list of objects in which case the result is either a recursively collected polynomial, or a polynomial in a distributed form with terms like $c*x1^e1*...*xn^en$, as specified by the `distributed` flag.

Note that the original polynomial needs to be in expanded form (for the variables concerned) in order for `collect()` to be able to find the coefficients properly.

The following `ginsh` transcript shows an application of `collect()` together with `find()`:

```
> a=expand((sin(x)+sin(y))*(1+p+q)*(1+d));
d*p*sin(x)+p*sin(x)+q*d*sin(x)+q*sin(y)+d*sin(x)+q*d*sin(y)+sin(y)+d*sin(y)
+q*sin(x)+d*sin(y)*p+sin(x)+sin(y)*p
> collect(a,{p,q});
d*sin(x)+(d*sin(x)+sin(y)+d*sin(y)+sin(x))*p
+(d*sin(x)+sin(y)+d*sin(y)+sin(x))*q+sin(y)+d*sin(y)+sin(x)
> collect(a,find(a,sin($1)));
(1+q+d+q*d+d*p+p)*sin(y)+(1+q+d+q*d+d*p+p)*sin(x)
> collect(a,{find(a,sin($1)),p,q});
(1+(1+d)*p+d+q*(1+d))*sin(x)+(1+(1+d)*p+d+q*(1+d))*sin(y)
> collect(a,{find(a,sin($1)),d});
(1+q+d*(1+q+p)+p)*sin(y)+(1+q+d*(1+q+p)+p)*sin(x)
```

Polynomials can often be brought into a more compact form by collecting common factors from the terms of sums. This is accomplished by the function

```
ex collect_common_factors(const ex & e);
```

This function doesn't perform a full factorization but only looks for factors which are already explicitly present:

```
> collect_common_factors(a*x+a*y);
(x+y)*a
> collect_common_factors(a*x^2+2*a*x*y+a*y^2);
a*(2*x*y+y^2+x^2)
> collect_common_factors(a*(b*(a+c)*x+b*((a+c)*x+(a+c)*y)*y));
(c+a)*a*(x*y+y^2+x)*b
```

5.7.3 Degree and coefficients

The degree and low degree of a polynomial in expanded form can be obtained using the two methods

```
int ex::degree(const ex & s);
int ex::ldegree(const ex & s);
```

These functions even work on rational functions, returning the asymptotic degree. By definition, the degree of zero is zero. To extract a coefficient with a certain power from an expanded polynomial you use

```
ex ex::coeff(const ex & s, int n);
```

You can also obtain the leading and trailing coefficients with the methods

```
ex ex::lcoeff(const ex & s);
ex ex::tcoeff(const ex & s);
```

which are equivalent to `coeff(s, degree(s))` and `coeff(s, ldegree(s))`, respectively.

An application is illustrated in the next example, where a multivariate polynomial is analyzed:

```
{
    symbol x("x"), y("y");
    ex PolyInp = 4*pow(x,3)*y + 5*x*pow(y,2) + 3*y
                 - pow(x+y,2) + 2*pow(y+2,2) - 8;
    ex Poly = PolyInp.expand();

    for (int i=Poly.ldegree(x); i<=Poly.degree(x); ++i) {
        cout << "The x^" << i << "-coefficient is "
             << Poly.coeff(x,i) << endl;
    }
    cout << "As polynomial in y: "
         << Poly.collect(y) << endl;
}
```

When run, it returns an output in the following fashion:

```
The x^0-coefficient is y^2+11*y
The x^1-coefficient is 5*y^2-2*y
The x^2-coefficient is -1
The x^3-coefficient is 4*y
As polynomial in y: -x^2+(5*x+1)*y^2+(-2*x+4*x^3+11)*y
```

As always, the exact output may vary between different versions of GiNaC or even from run to run since the internal canonical ordering is not within the user's sphere of influence.

`degree()`, `ldegree()`, `coeff()`, `lcoeff()`, `tcoeff()` and `collect()` can also be used to a certain degree with non-polynomial expressions as they not only work with symbols but with constants, functions and indexed objects as well:

```
{
    symbol a("a"), b("b"), c("c"), x("x");
    idx i(symbol("i"), 3);

    ex e = pow(sin(x) - cos(x), 4);
    cout << e.degree(cos(x)) << endl;
     // -> 4
    cout << e.expand().coeff(sin(x), 3) << endl;
     // -> -4*cos(x)

    e = indexed(a+b, i) * indexed(b+c, i);
    e = e.expand(expand_options::expand_indexed);
    cout << e.collect(indexed(b, i)) << endl;
     // -> a.i*c.i+(a.i+c.i)*b.i+b.i^2
}
```

5.7.4 Polynomial division

The two functions

```
ex quo(const ex & a, const ex & b, const ex & x);
ex rem(const ex & a, const ex & b, const ex & x);
```

compute the quotient and remainder of univariate polynomials in the variable '`x`'. The results satisfy $a = b * quo(a, b, x) + rem(a, b, x)$.

The additional function

```
ex prem(const ex & a, const ex & b, const ex & x);
```

computes the pseudo-remainder of '`a`' and '`b`' which satisfies $c * a = b * q + prem(a, b, x)$, where $c = b.lcoeff(x)^{(a.degree(x) - b.degree(x) + 1)}$.

Exact division of multivariate polynomials is performed by the function

```
bool divide(const ex & a, const ex & b, ex & q);
```

If '`b`' divides '`a`' over the rationals, this function returns `true` and returns the quotient in the variable `q`. Otherwise it returns `false` in which case the value of `q` is undefined.

5.7.5 Unit, content and primitive part

The methods

```
ex ex::unit(const ex & x);
ex ex::content(const ex & x);
ex ex::primpart(const ex & x);
ex ex::primpart(const ex & x, const ex & c);
```

return the unit part, content part, and primitive polynomial of a multivariate polynomial with respect to the variable '`x`' (the unit part being the sign of the leading coefficient, the content part being the GCD of the coefficients, and the primitive polynomial being the input polynomial divided by the unit and content parts). The second variant of `primpart()` expects the previously calculated content part of the polynomial in `c`, which enables it to work faster in the case where the content part has already been computed. The product of unit, content, and primitive part is the original polynomial.

Additionally, the method

```
void ex::unitcontprim(const ex & x, ex & u, ex & c, ex & p);
```

computes the unit, content, and primitive parts in one go, returning them in `u`, `c`, and `p`, respectively.

5.7.6 GCD, LCM and resultant

The functions for polynomial greatest common divisor and least common multiple have the synopsis

```
ex gcd(const ex & a, const ex & b);
ex lcm(const ex & a, const ex & b);
```

The functions `gcd()` and `lcm()` accept two expressions `a` and `b` as arguments and return a new expression, their greatest common divisor or least common multiple, respectively. If the polynomials `a` and `b` are coprime `gcd(a,b)` returns 1 and `lcm(a,b)` returns the product of `a` and `b`. Note that all the coefficients must be rationals.

```
#include <ginac/ginac.h>
using namespace GiNaC;

int main()
{
    symbol x("x"), y("y"), z("z");
    ex P_a = 4*x*y + x*z + 20*pow(y, 2) + 21*y*z + 4*pow(z, 2);
    ex P_b = x*y + 3*x*z + 5*pow(y, 2) + 19*y*z + 12*pow(z, 2);

    ex P_gcd = gcd(P_a, P_b);
    // x + 5*y + 4*z
    ex P_lcm = lcm(P_a, P_b);
    // 4*x*y^2 + 13*y*x*z + 20*y^3 + 81*y^2*z + 67*y*z^2 + 3*x*z^2 + 12*z^3
}
```

The resultant of two expressions only makes sense with polynomials. It is always computed with respect to a specific symbol within the expressions. The function has the interface

```
ex resultant(const ex & a, const ex & b, const ex & s);
```

Resultants are symmetric in `a` and `b`. The following example computes the resultant of two expressions with respect to `x` and `y`, respectively:

```
#include <ginac/ginac.h>
using namespace GiNaC;

int main()
{
    symbol x("x"), y("y");

    ex e1 = x+pow(y,2), e2 = 2*pow(x,3)-1; // x+y^2, 2*x^3-1
    ex r;

    r = resultant(e1, e2, x);
    // -> 1+2*y^6
    r = resultant(e1, e2, y);
    // -> 1-4*x^3+4*x^6
}
```

5.7.7 Square-free decomposition

Square-free decomposition is available in GiNaC:

```
ex sqrfree(const ex & a, const lst & l = lst{});
```

Here is an example that by the way illustrates how the exact form of the result may slightly depend on the order of differentiation, calling for some care with subsequent processing of the result:

```
...
symbol x("x"), y("y");
ex BiVarPol = expand(pow(2-2*y,3) * pow(1+x*y,2) * pow(x-2*y,2) * (x+y));

cout << sqrfree(BiVarPol, lst{x,y}) << endl;
 // -> 8*(1-y)^3*(y*x^2-2*y+x*(1-2*y^2))^2*(y+x)

cout << sqrfree(BiVarPol, lst{y,x}) << endl;
 // -> 8*(1-y)^3*(-y*x^2+2*y+x*(-1+2*y^2))^2*(y+x)

cout << sqrfree(BiVarPol) << endl;
 // -> depending on luck, any of the above
...
```

Note also, how factors with the same exponents are not fully factorized with this method.

5.7.8 Square-free partial fraction decomposition

GiNaC also supports square-free partial fraction decomposition of rational functions:

```
ex sqrfree_parfrac(const ex & a, const symbol & x);
```

It is called square-free because it assumes a square-free factorization of the input's denominator:

```
...
symbol x("x");

ex rat = (x-4)/(pow(x,2)*(x+2));
cout << sqrfree_parfrac(rat, x) << endl;
 // -> -2*x^(-2)+3/2*x^(-1)-3/2*(2+x)^(-1)
```

5.7.9 Polynomial factorization

Polynomials can also be fully factored with a call to the function

```
ex factor(const ex & a, unsigned int options = 0);
```

The factorization works for univariate and multivariate polynomials with rational coefficients. The following code snippet shows its capabilities:

```
...
cout << factor(pow(x,2)-1) << endl;
 // -> (1+x)*(-1+x)
cout << factor(expand((x-y*z)*(x-pow(y,2)-pow(z,3))*(x+y+z))) << endl;
 // -> (y+z+x)*(y*z-x)*(y^2-x+z^3)
cout << factor(pow(x,2)-1+sin(pow(x,2)-1)) << endl;
 // -> -1+sin(-1+x^2)+x^2
...
```

The results are as expected except for the last one where no factorization seems to have been done. This is due to the default option `factor_options::polynomial` (equals zero) to `factor()`, which tells GiNaC to try a factorization only if the expression is a valid polynomial. In the shown example this is not the case, because one term is a function.

There exists a second option `factor_options::all`, which tells GiNaC to ignore non-polynomial parts of an expression and also to look inside function arguments. With this option the example gives:

```
...
```

```
cout << factor(pow(x,2)-1+sin(pow(x,2)-1), factor_options::all)
     << endl;
 // -> (-1+x)*(1+x)+sin((-1+x)*(1+x))
...
```

GiNaC's factorization functions cannot handle algebraic extensions. Therefore the following example does not factor:

```
...
cout << factor(pow(x,2)-2) << endl;
 // -> -2+x^2  and not  (x-sqrt(2))*(x+sqrt(2))
...
```

Factorization is useful in many applications. A lot of algorithms in computer algebra depend on the ability to factor a polynomial. Of course, factorization can also be used to simplify expressions, but it is costly and applying it to complicated expressions (high degrees or many terms) may consume far too much time. So usually, looking for a GCD at strategic points in a calculation is the cheaper and more appropriate alternative.

5.8 Rational expressions

5.8.1 The `normal` method

Some basic form of simplification of expressions is called for frequently. GiNaC provides the method `.normal()`, which converts a rational function into an equivalent rational function of the form '`numerator/denominator`' where numerator and denominator are coprime. If the input expression is already a fraction, it just finds the GCD of numerator and denominator and cancels it, otherwise it performs fraction addition and multiplication.

`.normal()` can also be used on expressions which are not rational functions as it will replace all non-rational objects (like functions or non-integer powers) by temporary symbols to bring the expression to the domain of rational functions before performing the normalization, and re-substituting these symbols afterwards. This algorithm is also available as a separate method `.to_rational()`, described below.

This means that both expressions `t1` and `t2` are indeed simplified in this little code snippet:

```
{
    symbol x("x");
    ex t1 = (pow(x,2) + 2*x + 1)/(x + 1);
    ex t2 = (pow(sin(x),2) + 2*sin(x) + 1)/(sin(x) + 1);
    std::cout << "t1 is " << t1.normal() << std::endl;
    std::cout << "t2 is " << t2.normal() << std::endl;
}
```

Of course this works for multivariate polynomials too, so the ratio of the sample-polynomials from the section about GCD and LCM above would be normalized to `P_a/P_b` = `(4*y+z)/(y+3*z)`.

5.8.2 Numerator and denominator

The numerator and denominator of an expression can be obtained with

```
ex ex::numer();
ex ex::denom();
ex ex::numer_denom();
```

These functions will first normalize the expression as described above and then return the numerator, denominator, or both as a list, respectively. If you need both numerator and denominator, call `numer_denom()`: it is faster than using `numer()` and `denom()` separately. And even more

important: a separate evaluation of `numer()` and `denom()` may result in a spurious sign, e.g. for $x/(x^2-1)$ `numer()` may return x and `denom()` $1-x^2$.

5.8.3 Converting to a polynomial or rational expression

Some of the methods described so far only work on polynomials or rational functions. GiNaC provides a way to extend the domain of these functions to general expressions by using the temporary replacement algorithm described above. You do this by calling

```
ex ex::to_polynomial(exmap & m);
```

or

```
ex ex::to_rational(exmap & m);
```

on the expression to be converted. The supplied `exmap` will be filled with the generated temporary symbols and their replacement expressions in a format that can be used directly for the `subs()` method. It can also already contain a list of replacements from an earlier application of `.to_polynomial()` or `.to_rational()`, so it's possible to use it on multiple expressions and get consistent results.

The difference between `.to_polynomial()` and `.to_rational()` is probably best illustrated with an example:

```
{
    symbol x("x"), y("y");
    ex a = 2*x/sin(x) - y/(3*sin(x));
    cout << a << endl;

    exmap mp;
    ex p = a.to_polynomial(mp);
    cout << " = " << p << "\n   with " << mp << endl;
     // = symbol3*symbol2*y+2*symbol2*x
     //   with {symbol2==sin(x)^(-1),symbol3==-1/3}

    exmap mr;
    ex r = a.to_rational(mr);
    cout << " = " << r << "\n   with " << mr << endl;
     // = -1/3*symbol4^(-1)*y+2*symbol4^(-1)*x
     //   with {symbol4==sin(x)}
}
```

The following more useful example will print '`sin(x)-cos(x)`':

```
{
    symbol x("x");
    ex a = pow(sin(x), 2) - pow(cos(x), 2);
    ex b = sin(x) + cos(x);
    ex q;
    exmap m;
    divide(a.to_polynomial(m), b.to_polynomial(m), q);
    cout << q.subs(m) << endl;
}
```

5.9 Symbolic differentiation

GiNaC's objects know how to differentiate themselves. Thus, a polynomial (class `add`) knows that its derivative is the sum of the derivatives of all the monomials:

```
{
```

```
    symbol x("x"), y("y"), z("z");
    ex P = pow(x, 5) + pow(x, 2) + y;

    cout << P.diff(x,2) << endl;
     // -> 20*x^3 + 2
    cout << P.diff(y) << endl;    // 1
     // -> 1
    cout << P.diff(z) << endl;    // 0
     // -> 0
}
```

If a second integer parameter *n* is given, the `diff` method returns the *n*th derivative.

If *every* object and every function is told what its derivative is, all derivatives of composed objects can be calculated using the chain rule and the product rule. Consider, for instance the expression `1/cosh(x)`. Since the derivative of `cosh(x)` is `sinh(x)` and the derivative of `pow(x,-1)` is `-pow(x,-2)`, GiNaC can readily compute the composition. It turns out that the composition is the generating function for Euler Numbers, i.e. the so called *n*th Euler number is the coefficient of `x^n/n!` in the expansion of `1/cosh(x)`. We may use this identity to code a function that generates Euler numbers in just three lines:

```
#include <ginac/ginac.h>
using namespace GiNaC;

ex EulerNumber(unsigned n)
{
    symbol x;
    const ex generator = pow(cosh(x),-1);
    return generator.diff(x,n).subs(x==0);
}

int main()
{
    for (unsigned i=0; i<11; i+=2)
        std::cout << EulerNumber(i) << std::endl;
    return 0;
}
```

When you run it, it produces the sequence `1`, `-1`, `5`, `-61`, `1385`, `-50521`. We increment the loop variable `i` by two since all odd Euler numbers vanish anyways.

5.10 Series expansion

Expressions know how to expand themselves as a Taylor series or (more generally) a Laurent series. As in most conventional Computer Algebra Systems, no distinction is made between those two. There is a class of its own for storing such series (`class pseries`) and a built-in function (called `Order`) for storing the order term of the series. As a consequence, if you want to work with series, i.e. multiply two series, you need to call the method `ex::series` again to convert it to a series object with the usual structure (expansion plus order term). A sample application from special relativity could read:

```
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

int main()
```

```
{
    symbol v("v"), c("c");

    ex gamma = 1/sqrt(1 - pow(v/c,2));
    ex mass_nonrel = gamma.series(v==0, 10);

    cout << "the relativistic mass increase with v is " << endl
         << mass_nonrel << endl;

    cout << "the inverse square of this series is " << endl
         << pow(mass_nonrel,-2).series(v==0, 10) << endl;
}
```

Only calling the series method makes the last output simplify to $1 - v^2/c^2 + O(v^10)$, without that call we would just have a long series raised to the power -2.

As another instructive application, let us calculate the numerical value of Archimedes' constant π (for which there already exists the built-in constant `Pi`) using John Machin's amazing formula $\pi = 16 \text{ atan } (\frac{1}{5}) - 4 \text{ atan } (\frac{1}{239})$. This equation (and similar ones) were used for over 200 years for computing digits of pi (see *Pi Unleashed*). We may expand the arcus tangent around `0` and insert the fractions `1/5` and `1/239`. However, as we have seen, a series in GiNaC carries an order term with it and the question arises what the system is supposed to do when the fractions are plugged into that order term. The solution is to use the function `series_to_poly()` to simply strip the order term off:

```
#include <ginac/ginac.h>
using namespace GiNaC;

ex machin_pi(int degr)
{
    symbol x;
    ex pi_expansion = series_to_poly(atan(x).series(x==0,degr));
    ex pi_approx = 16*pi_expansion.subs(x==numeric(1,5))
                   -4*pi_expansion.subs(x==numeric(1,239));
    return pi_approx;
}

int main()
{
    using std::cout;  // just for fun, another way of...
    using std::endl;  // ...dealing with this namespace std.
    ex pi_frac;
    for (int i=2; i<12; i+=2) {
        pi_frac = machin_pi(i);
        cout << i << ":\t" << pi_frac << endl
             << "\t" << pi_frac.evalf() << endl;
    }
    return 0;
}
```

Note how we just called `.series(x,degr)` instead of `.series(x==0,degr)`. This is a simple shortcut for `ex`'s method `series()`: if the first argument is a symbol the expression is expanded in that symbol around point `0`. When you run this program, it will type out:

```
2:      3804/1195
        3.1832635983263598326
```

```
4:      5359397032/1706489875
        3.1405970293260603143
6:      38279241713339684/12184551018734375
        3.141621029325034425
8:      765284871091801925409760/24359780855939418203125
        3.141591772182177295
10:     327853873402258685803048818236/104359128170408663038552734375
        3.1415926824043995174
```

5.11 Symmetrization

The three methods

```
ex ex::symmetrize(const lst & l);
ex ex::antisymmetrize(const lst & l);
ex ex::symmetrize_cyclic(const lst & l);
```

symmetrize an expression by returning the sum over all symmetric, antisymmetric or cyclic permutations of the specified list of objects, weighted by the number of permutations.

The three additional methods

```
ex ex::symmetrize();
ex ex::antisymmetrize();
ex ex::symmetrize_cyclic();
```

symmetrize or antisymmetrize an expression over its free indices.

Symmetrization is most useful with indexed expressions but can be used with almost any kind of object (anything that is `subs()`able):

```
{
    idx i(symbol("i"), 3), j(symbol("j"), 3), k(symbol("k"), 3);
    symbol A("A"), B("B"), a("a"), b("b"), c("c");

    cout << ex(indexed(A, i, j)).symmetrize() << endl;
     // -> 1/2*A.j.i+1/2*A.i.j
    cout << ex(indexed(A, i, j, k)).antisymmetrize(lst{i, j}) << endl;
     // -> -1/2*A.j.i.k+1/2*A.i.j.k
    cout << ex(lst{a, b, c}).symmetrize_cyclic(lst{a, b, c}) << endl;
     // -> 1/3*{a,b,c}+1/3*{b,c,a}+1/3*{c,a,b}
}
```

5.12 Predefined mathematical functions

5.12.1 Overview

GiNaC contains the following predefined mathematical functions:

Name	Function
`abs(x)`	absolute value
`step(x)`	step function
`csgn(x)`	complex sign
`conjugate(x)`	complex conjugation
`real_part(x)`	real part
`imag_part(x)`	imaginary part
`sqrt(x)`	square root (not a GiNaC function, rather an alias for `pow(x, numeric(1, 2))`)
`sin(x)`	sine
`cos(x)`	cosine
`tan(x)`	tangent
`asin(x)`	inverse sine
`acos(x)`	inverse cosine
`atan(x)`	inverse tangent
`atan2(y, x)`	inverse tangent with two arguments
`sinh(x)`	hyperbolic sine
`cosh(x)`	hyperbolic cosine
`tanh(x)`	hyperbolic tangent
`asinh(x)`	inverse hyperbolic sine
`acosh(x)`	inverse hyperbolic cosine
`atanh(x)`	inverse hyperbolic tangent
`exp(x)`	exponential function
`log(x)`	natural logarithm
`eta(x,y)`	Eta function: `eta(x,y) = log(x*y) - log(x) - log(y)`
`Li2(x)`	dilogarithm
`Li(m, x)`	classical polylogarithm as well as multiple polylogarithm
`G(a, y)`	multiple polylogarithm
`G(a, s, y)`	multiple polylogarithm with explicit signs for the imaginary parts
`S(n, p, x)`	Nielsen's generalized polylogarithm
`H(m, x)`	harmonic polylogarithm
`zeta(m)`	Riemann's zeta function as well as multiple zeta value
`zeta(m, s)`	alternating Euler sum
`zetaderiv(n, x)`	derivatives of Riemann's zeta function
`Gt(a, y, tau)`	elliptic multiple polylogarithm
`iterated_integral(a, y)`	iterated integral
`iterated_integral(a, y, N)`	iterated integral with explicit truncation parameter
`tgamma(x)`	gamma function
`lgamma(x)`	logarithm of gamma function
`beta(x, y)`	beta function (`tgamma(x)*tgamma(y)/tgamma(x+y)`)
`psi(x)`	psi (digamma) function
`psi(n, x)`	derivatives of psi function (polygamma functions)
`EllipticK(x)`	complete elliptic integral of the first kind
`EllipticE(x)`	complete elliptic integral of the second kind
`factorial(n)`	factorial function $n!$
`binomial(n, k)`	binomial coefficients
`Order(x)`	order term function in truncated power series

For functions that have a branch cut in the complex plane, GiNaC follows the conventions of C/C++ for systems that do not support a signed zero. In particular: the natural logarithm (`log`) and the square root (`sqrt`) both have their branch cuts running along the negative real axis. The `asin`, `acos`, and `atanh` functions all have two branch cuts starting at +/-1 and running away towards infinity along the real axis. The `atan` and `asinh` functions have two branch cuts starting at +/-i and running away towards infinity along the imaginary axis. The `acosh` function has one branch cut starting at +1 and running towards -infinity. These functions are continuous as the branch cut is approached coming around the finite endpoint of the cut in a counter clockwise direction.

5.12.2 Expanding functions

GiNaC knows several expansion laws for trancedent functions, e.g. $e^{a+b} = e^a e^b$, $|zw| = |z| \cdot |w|$ or $\log(c * d) = \log(c) + \log(d)$, (for positive c, d). In order to use these rules you need to call `expand()` method with the option `expand_options::expand_transcendental`. Another relevant option is `expand_options::expand_function_args`. Their usage and interaction can be seen from the following example:

```
{
symbol x("x"),  y("y");
ex e=exp(pow(x+y,2));
cout << e.expand() << endl;
// -> exp((x+y)^2)
cout << e.expand(expand_options::expand_transcendental) << endl;
// -> exp((x+y)^2)
cout << e.expand(expand_options::expand_function_args) << endl;
// -> exp(2*x*y+x^2+y^2)
cout << e.expand(expand_options::expand_function_args
| expand_options::expand_transcendental) << endl;
// -> exp(y^2)*exp(2*x*y)*exp(x^2)
}
```

If both flags are set (as in the last call), then GiNaC tries to get the maximal expansion. For example, for the exponent GiNaC firstly expands the argument and then the function. For the logarithm and absolute value, GiNaC uses the opposite order: firstly expands the function and then its argument. Of course, a user can fine-tune this behavior by sequential calls of several `expand()` methods with desired flags.

5.12.3 Multiple polylogarithms

The multiple polylogarithm is the most generic member of a family of functions, to which others like the harmonic polylogarithm, Nielsen's generalized polylogarithm and the multiple zeta value belong. Each of these functions can also be written as a multiple polylogarithm with specific parameters. This whole family of functions is therefore often referred to simply as multiple polylogarithms, containing `Li`, `G`, `H`, `S` and `zeta`. The multiple polylogarithm itself comes in two variants: `Li` and `G`. While `Li` and `G` in principle represent the same function, the different notations are more natural to the series representation or the integral representation, respectively.

To facilitate the discussion of these functions we distinguish between indices and arguments as parameters. In the table above indices are printed as `m`, `s`, `n` or `p`, whereas arguments are printed as `x`, `a` and `y`.

To define a `Li`, `H` or `zeta` with a depth greater than one, you have to pass a GiNaC `lst` for the indices `m` and `s`, and in the case of `Li` for the argument `x` as well. The parameter `a` of `G` must always be a `lst` containing the arguments in expanded form. If `G` is used with a third parameter `s`, `s` must have the same length as `a`. It contains then the signs of the imaginary parts of the

arguments. If `s` is not given, the signs default to `+1`. Note that `Li` and `zeta` are polymorphic in this respect. They can stand in for the classical polylogarithm and Riemann's zeta function (if depth is one), as well as for the multiple polylogarithm and the multiple zeta value, respectively. Note also, that GiNaC doesn't check whether the `lst`s for two parameters do have the same length. It is up to the user to ensure this, otherwise evaluating will result in undefined behavior.

The functions print in LaTeX format as $\mathrm{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k)$, $\mathrm{S}_{n,p}(x)$, $\mathrm{H}_{m_1,m_2,\ldots,m_k}(x)$ and $\zeta(m_1,m_2,\ldots,m_k)$. If `zeta` is an alternating zeta sum, i.e. `zeta(m,s)`, the indices with negative sign are printed with a line above, e.g. $\zeta(5,\overline{2})$. The order of indices and arguments in the GiNaC `lst`s and in the output is the same.

Definitions and analytical as well as numerical properties of multiple polylogarithms are too numerous to be covered here. Instead, the user is referred to the publications listed at the end of this section. The implementation in GiNaC adheres to the definitions and conventions therein, except for a few differences which will be explicitly stated in the following.

One difference is about the order of the indices and arguments. For GiNaC we adopt the convention that the indices and arguments are understood to be in the same order as in which they appear in the series representation. This means $\mathrm{Li}_{m_1,m_2,m_3}(x,1,1) = \mathrm{H}_{m_1,m_2,m_3}(x)$ and $\mathrm{Li}_{2,1}(1,1) = \zeta(2,1) = \zeta(3)$, but $\zeta(1,2)$ evaluates to infinity. So in comparison to the older ones of the referenced publications the order of indices and arguments for `Li` is reversed.

The functions only evaluate if the indices are integers greater than zero, except for the indices `s` in `zeta` and `G` as well as `m` in `H`. Since `s` will be interpreted as the sequence of signs for the corresponding indices `m` or the sign of the imaginary part for the corresponding arguments `a`, it must contain 1 or -1, e.g. `zeta(lst{3,4}, lst{-1,1})` means $\zeta(\overline{3},4)$ and `G(lst{a,b}, lst{-1,1}, c)` means $G(a-0\epsilon, b+0\epsilon; c)$. The definition of `H` allows indices to be 0, 1 or -1 (in expanded notation) or equally to be any integer (in compact notation). With GiNaC expanded and compact notation can be mixed, e.g. `lst{0,0,-1,0,1,0,0}`, `lst{0,0,-1,2,0,0}` and `lst{-3,2,0,0}` are equivalent as indices. The anonymous evaluator `eval()` tries to reduce the functions, if possible, to the least-generic multiple polylogarithm. If all arguments are unit, it returns `zeta`. Arguments equal to zero get considered, too. Riemann's zeta function `zeta` (with depth one) evaluates also for negative integers and positive even integers. For example:

```
> Li({3,1},{x,1});
S(2,2,x)
> H({-3,2},1);
-zeta({3,2},{-1,-1})
> S(3,1,1);
1/90*Pi^4
```

It is easy to tell for a given function into which other function it can be rewritten, may it be a less-generic or a more-generic one, except for harmonic polylogarithms `H` with negative indices or trailing zeros (the example above gives a hint). Signs can quickly be messed up, for example. Therefore GiNaC offers a C++ function `convert_H_to_Li()` to deal with the upgrade of a `H` to a multiple polylogarithm `Li` (`eval()` already cares for the possible downgrade):

```
> convert_H_to_Li({0,-2,-1,3},x);
Li({3,1,3},{-x,1,-1})
> convert_H_to_Li({2,-1,0},x);
-Li({2,1},{x,-1})*log(x)+2*Li({3,1},{x,-1})+Li({2,2},{x,-1})
```

Every function can be numerically evaluated for arbitrary real or complex arguments. The precision is arbitrary and can be set through the global variable `Digits`:

```
> Digits=100;
100
> evalf(zeta({3,1,3,1}));
0.00522956956353096010093065228389923158989042078463463552254744897214886954 4...
```

Note that the convention for arguments on the branch cut in GiNaC as stated above is different from the one Remiddi and Vermaseren have chosen for the harmonic polylogarithm.

If a function evaluates to infinity, no exceptions are raised, but the function is returned unevaluated, e.g. $\zeta(1)$. In long expressions this helps a lot with debugging, because you can easily spot the divergencies. But on the other hand, you have to make sure for yourself, that no illegal cancellations of divergencies happen.

Useful publications:

Nested Sums, Expansion of Transcendental Functions and Multi-Scale Multi-Loop Integrals, S.Moch, P.Uwer, S.Weinzierl, hep-ph/0110083

Harmonic Polylogarithms, E.Remiddi, J.A.M.Vermaseren, Int.J.Mod.Phys. A15 (2000), pp. 725-754

Special Values of Multiple Polylogarithms, J.Borwein, D.Bradley, D.Broadhurst, P.Lisonek, Trans.Amer.Math.Soc. 353/3 (2001), pp. 907-941

Numerical Evaluation of Multiple Polylogarithms, J.Vollinga, S.Weinzierl, hep-ph/0410259

5.12.4 Elliptic multiple polylogarithms

A general elliptic multiple polylogarithm $\tilde{\Gamma}\left((n_1, z_1), \ldots, (n_k, z_k); z, \tau\right)$ is represented in GiNaC as

```
Gt({{n1,z1},...,{nk,zk}},z,tau).
```

The `ni` should be non-negative integers, while all other arguments may be arbitrary symbolic expressions. For performance reasons, the class `Gt` is implemented as a derived class of class `basic` and not as an object of the class `function`. As a consequence, the methods `op`, `let_op` and `match`, which are available for `function`s are not supported for `Gt`s. Instead, read-only access to the arguments is provided via the methods `getZ`, `getTau` and `getArgs`.

If all arguments can be directly evaluated to numerical values, a `Gt` object may be evaluated using the `evalf` function. When compiled and executed, the following program

```
#include <iostream>
#include <ginac/ginac.h>
using namespace GiNaC;

int main() {
    ex expr = Gt(lst{lst{2,3-2*I/7},lst{1,-2+I/7}},1+I,2+3*I);
    std::cout << expr << " = " << expr.evalf() << std::endl;

    return 0;
}
```

prints

```
Gt({{2,3-2/7*I},{1,-2+1/7*I}},1+I,2+3*I)
= -0.24570758059458382605-7.8354094938915607015*I.
```

Expressions containing multiple eMPLs can be evaluated using the `Gt::ex_evaluate` function. This function is usually preferable over `evalf`, as it applies the algorithm to all eMPLs simultaneously. This enables reusing partial results and allows cancellations of intermediate terms. This function also takes a list of replacement rules as an optional second argument. These replacements are applied to all `Gt` arguments prior to numeric evaluation, which allows keeping arguments symbolic for as long as possible. The syntax is illustrated in the following example:

```
symbol z{"z"}, tau{"tau"};
ex expr = Gt(lst{lst{1,numeric(1,6)+I/6}},z,tau)
        + 12*Pi*Gt(lst{lst{1,numeric(5,6)+I/6}},z,tau);
ex replacements = lst{z == numeric(1,3)+I/5, tau == numeric(2,5)+I/3};
```

```
std::cout << expr << " = "
          << Gt::ex_evaluate(expr, replacements) << std::endl;
```

The function `Gt::lst_evaluate` works similarly, except that its first argument is a list of expressions and it returns a list of numeric values. This allows reusing partial results across the multiple expressions.

For advanced usage, we also provide the functions `Gt::ex_zisToFundamental`, `Gt::ex_regularise`, `Gt::ex_tauToFundamental` and `Gt::ex_cutIntegrationPath` which perform the individual steps of the algorithm separately, not including possible repetitions of earlier steps. We also provide `Gt::ex_prepare`, which performs all preparation steps, and `Gt::ex_qExpand`, which performs the q-expansion on fully prepared eMPLs. All of these functions take an expression as a first argument, optionally a replacement list as a second argument and return a transformed expression.

Usage in `ginsh`: The syntax for defining eMPLs in `ginsh` is identical to the one described above. The evaluation is performed using the function `Gt_evaluate`, which corresponds to both the `Gt::ex_evaluate` and `Gt::lst_evaluate` functions. Since functions in `ginsh` do not have a definite return type, the distinction between `Gt::ex_evaluate` and `Gt::lst_evaluate` is not necessary. The function `Gt_evaluate` always requires a (possibly empty) replacement list as a second argument. Interactive use of the package is illustrated in the following example:

```
> z = 1+I;
> tau = 2+3*I;
> expr = Gt({{2,z1},{1,z2}},z,tau);
> Gt_evaluate(expr, {z1 == 3-2*I/7, z2 ==-2+I/7});
-0.24570758059458382562-7.835409493891560702*I
```

Specifying deformations around poles: It may be necessary to specify deformations of the integration contour around poles in the integrand. Deformations can be specified as an optional third argument in the definition of each integration kernel $g^{(n)}(z,\tau)$, i.e., $\{n,z,\Delta\}$ where $\Delta = \pm 1$. If the locus z becomes singular and the integration contour passes over it, the path is deformed around z, as specified by Δ. A value of $\Delta = +1$ indicates that the path should pass the pole on the right in the direction of integration, i.e., the path is deformed over the point $z\exp(-i\epsilon\Delta)$ for some appropriate value of $\epsilon > 0$. For example, if the integration runs along the positive real axis, a value of $\Delta = +1$ leads to a deformation toward negative imaginary values. This convention is a natural extension of the corresponding convention for ordinary MPLs in GiNaC. If no deformation is specified, the default value $\Delta = +1$ is used.

The complete documentation can be found in the publication:

Elliptic Multiple Polylogarithms in GiNaC, C. Duhr, F. Lorkowski, R. Marzucca, S. Mauch, S. Weinzierl, arXiv:2602.09956

5.12.5 Iterated integrals

Multiple polylogarithms are a particular example of iterated integrals. An iterated integral is defined by the function `iterated_integral(a,y)`. The variable `y` gives the upper integration limit for the outermost integration, by convention the lower integration limit is always set to zero. The variable `a` must be a GiNaC `lst` containing sub-classes of `integration_kernel` as elements. The depth of the iterated integral corresponds to the number of elements of `a`. The available integrands for iterated integrals are (for a more detailed description the user is referred to the publications listed at the end of this section)

Class	Description
`integration_kernel()`	Base class, represents the one-form dy
`basic_log_kernel()`	Logarithmic one-form dy/y
`multiple_polylog_kernel(z_j)`	The one-form $dy/(y-z_j)$
`ELi_kernel(n, m, x, y)`	The one form $ELi_{n;m}(x;y;q)dq/q$
`Ebar_kernel(n, m, x, y)`	The one form $\overline{E}_{n;m}(x;y;q)dq/q$
`Kronecker_dtau_kernel(k, z_j, K, C_k)`	The one form $C_k K(k-1)/(2\pi i)^k g^{(k)}(z_j, K\tau)dq/q$
`Kronecker_dz_kernel(k, z_j, tau, K, C_k)`	The one form $C_k(2\pi i)^{2-k} g^{(k-1)}(z-z_j, K\tau)dz$
`Eisenstein_kernel(k, N, a, b, K, C_k)`	The one form $C_k E_{k,N,a,b,K}(\tau)dq/q$
`Eisenstein_h_kernel(k, N, r, s, C_k)`	The one form $C_k h_{k,N,r,s}(\tau)dq/q$
`modular_form_kernel(k, P, C_k)`	The one form $C_k P dq/q$
`user_defined_kernel(f, y)`	The one form $f(y)dy$

All parameters are assumed to be such that all integration kernels have a convergent Laurent expansion around zero with at most a simple pole at zero. The iterated integral may also be called with an optional third parameter `iterated_integral(a,y,N_trunc)`, in which case the numerical evaluation will truncate the series expansion at order `N_trunc`.

The classes `Eisenstein_kernel()`, `Eisenstein_h_kernel()` and `modular_form_kernel()` provide a method `q_expansion_modular_form(q, order)`, which can used to obtain the q-expansion of $E_{k,N,a,b,K}(\tau)$, $h_{k,N,r,s}(\tau)$ or P to the specified order.

Useful publications:

Numerical evaluation of iterated integrals related to elliptic Feynman integrals, M.Walden, S.Weinzierl, arXiv:2010.05271

5.13 Complex expressions

For dealing with complex expressions there are the methods

```
ex ex::conjugate();
ex ex::real_part();
ex ex::imag_part();
```

that return respectively the complex conjugate, the real part and the imaginary part of an expression. Complex conjugation works as expected for all built-in functions and objects. Taking real and imaginary parts has not yet been implemented for all built-in functions. In cases where it is not known how to conjugate or take a real/imaginary part one of the functions `conjugate`, `real_part` or `imag_part` is returned. For instance, in case of a complex symbol `x` (symbols are complex by default), one could not simplify `conjugate(x)`. In the case of strings of gamma matrices, the `conjugate` method takes the Dirac conjugate.

For example,

```
{
    varidx a(symbol("a"), 4), b(symbol("b"), 4);
    symbol x("x");
    realsymbol y("y");

    cout << (3*I*x*y + sin(2*Pi*I*y)).conjugate() << endl;
```

```
    // -> -3*I*conjugate(x)*y+sin(-2*I*Pi*y)
    cout << (dirac_gamma(a)*dirac_gamma(b)*dirac_gamma5()).conjugate() << endl;
    // -> -gamma5*gamma~b*gamma~a
}
```

If you declare your own GiNaC functions and you want to conjugate them, you will have to supply a specialized conjugation method for them (see Section 6.2 [Symbolic functions], page 91, and the GiNaC source-code for `abs` as an example). GiNaC does not automatically conjugate user-supplied functions by conjugating their arguments because this would be incorrect on branch cuts. Also, specialized methods can be provided to take real and imaginary parts of user-defined functions.

5.14 Solving linear systems of equations

The function `lsolve()` provides a convenient wrapper around some matrix operations that comes in handy when a system of linear equations needs to be solved:

```
ex lsolve(const ex & eqns, const ex & symbols,
          unsigned options = solve_algo::automatic);
```

Here, `eqns` is a `lst` of equalities (i.e. class `relational`) while `symbols` is a `lst` of indeterminates. (See Section 4.4 [The class hierarchy], page 12, for an exposition of class `lst`).

It returns the `lst` of solutions as an expression. As an example, let us solve the two equations `a*x+b*y==3` and `x-y==b`:

```
{
    symbol a("a"), b("b"), x("x"), y("y");
    lst eqns = {a*x+b*y==3, x-y==b};
    lst vars = {x, y};
    cout << lsolve(eqns, vars) << endl;
    // -> {x==(3+b^2)/(b+a),y==(3-b*a)/(b+a)}
```

When the linear equations `eqns` are underdetermined, the solution will contain one or more tautological entries like `x==x`, depending on the rank of the system. When they are overdetermined, the solution will be an empty `lst`. Note the third optional parameter to `lsolve()`: it accepts the same parameters as `matrix::solve()`. This is because `lsolve` is just a wrapper around that method.

5.15 Input and output of expressions

5.15.1 Expression output

Expressions can simply be written to any stream:

```
{
    symbol x("x");
    ex e = 4.5*I+pow(x,2)*3/2;
    cout << e << endl;    // prints '4.5*I+3/2*x^2'
    // ...
```

The default output format is identical to the `ginsh` input syntax and to that used by most computer algebra systems, but not directly pastable into a GiNaC C++ program (note that in the above example, `pow(x,2)` is printed as '`x^2`').

It is possible to print expressions in a number of different formats with a set of stream manipulators;

```
std::ostream & dflt(std::ostream & os);
std::ostream & latex(std::ostream & os);
```

```
std::ostream & tree(std::ostream & os);
std::ostream & csrc(std::ostream & os);
std::ostream & csrc_float(std::ostream & os);
std::ostream & csrc_double(std::ostream & os);
std::ostream & csrc_cl_N(std::ostream & os);
std::ostream & index_dimensions(std::ostream & os);
std::ostream & no_index_dimensions(std::ostream & os);
```

The `tree`, `latex` and `csrc` formats are also available in `ginsh` via the `print()`, `print_latex()` and `print_csrc()` functions, respectively.

All manipulators affect the stream state permanently. To reset the output format to the default, use the `dflt` manipulator:

```
// ...
cout << latex;            // all output to cout will be in LaTeX format from
                          // now on
cout << e << endl;        // prints '4.5 i+\frac{3}{2} x^{2}'
cout << sin(x/2) << endl; // prints '\sin(\frac{1}{2} x)'
cout << dflt;             // revert to default output format
cout << e << endl;        // prints '4.5*I+3/2*x^2'
// ...
```

If you don't want to affect the format of the stream you're working with, you can output to a temporary `ostringstream` like this:

```
// ...
ostringstream s;
s << latex << e;         // format of cout remains unchanged
cout << s.str() << endl; // prints '4.5 i+\frac{3}{2} x^{2}'
// ...
```

The `csrc` (an alias for `csrc_double`), `csrc_float`, `csrc_double` and `csrc_cl_N` manipulators set the output to a format that can be directly used in a C or C++ program. The three possible formats select the data types used for numbers (`csrc_cl_N` uses the classes provided by the CLN library):

```
// ...
cout << "f = " << csrc_float << e << ";\n";
cout << "d = " << csrc_double << e << ";\n";
cout << "n = " << csrc_cl_N << e << ";\n";
// ...
```

The above example will produce (note the `x^2` being converted to `x*x`):

```
f = (3.0/2.0)*(x*x)+std::complex<float>(0.0,4.5000000e+00);
d = (3.0/2.0)*(x*x)+std::complex<double>(0.0,4.5000000000000000e+00);
n = cln::cl_RA("3/2")*(x*x)+cln::complex(cln::cl_I("0"),cln::cl_F("4.5_17"));
```

The `tree` manipulator allows dumping the internal structure of an expression for debugging purposes:

```
// ...
cout << tree << e;
}
```

produces

```
add, hash=0x0, flags=0x3, nops=2
    power, hash=0x0, flags=0x3, nops=2
        x (symbol), serial=0, hash=0xc8d5bcdd, flags=0xf
        2 (numeric), hash=0x6526b0fa, flags=0xf
```

```
3/2 (numeric), hash=0xf9828fbd, flags=0xf
-----
overall_coeff
4.5L0i (numeric), hash=0xa40a97e0, flags=0xf
=====
```

The `latex` output format is for LaTeX parsing in mathematical mode. It is rather similar to the default format but provides some braces needed by LaTeX for delimiting boxes and also converts some common objects to conventional LaTeX names. It is possible to give symbols a special name for LaTeX output by supplying it as a second argument to the `symbol` constructor. For example, the code snippet

```
{
    symbol x("x", "\\circ");
    ex e = lgamma(x).series(x==0,3);
    cout << latex << e << endl;
}
```

will print

```
{(-\ln(\circ))}+{(-\gamma_E)} \circ+{(\frac{1}{12} \pi^{2})} \circ^{2}
+\mathcal{O}(\circ^{3})
```

Index dimensions are normally hidden in the output. To make them visible, use the `index_dimensions` manipulator. The dimensions will be written in square brackets behind each index value in the default and LaTeX output formats:

```
{
    symbol x("x"), y("y");
    varidx mu(symbol("mu"), 4), nu(symbol("nu"), 4);
    ex e = indexed(x, mu) * indexed(y, nu);

    cout << e << endl;
     // prints 'x~mu*y~nu'
    cout << index_dimensions << e << endl;
     // prints 'x~mu[4]*y~nu[4]'
    cout << no_index_dimensions << e << endl;
     // prints 'x~mu*y~nu'
}
```

If you need any fancy special output format, e.g. for interfacing GiNaC with other algebra systems or for producing code for different programming languages, you can always traverse the expression tree yourself:

```
static void my_print(const ex & e)
{
    if (is_a<GiNaC::function>(e))
        cout << ex_to<GiNaC::function>(e).get_name();
    else
        cout << ex_to<basic>(e).class_name();
    cout << "(";
    size_t n = e.nops();
    if (n)
        for (size_t i=0; i<n; i++) {
            my_print(e.op(i));
            if (i != n-1)
                cout << ",";
        }
```

```
        else
            cout << e;
        cout << ")";
    }

    int main()
    {
        symbol x("x"), y("y");
        my_print(pow(3, x) - 2 * sin(y / Pi));
        cout << endl;
        return 0;
    }
```

(Note that we have added the namespace resolution to the `GiNaC::function` class to avoid modern compilers' confusion with `std::function`.)

This will produce

```
add(power(numeric(3),symbol(x)),mul(sin(mul(power(constant(Pi),numeric(-1)),
symbol(y))),numeric(-2)))
```

If you need an output format that makes it possible to accurately reconstruct an expression by feeding the output to a suitable parser or object factory, you should consider storing the expression in an `archive` object and reading the object properties from there. See the section on archiving for more information.

5.15.2 Expression input

GiNaC provides no way to directly read an expression from a stream because you will usually want the user to be able to enter something like '`2*x+sin(y)`' and have the '`x`' and '`y`' correspond to the symbols `x` and `y` you defined in your program and there is no way to specify the desired symbols to the `>>` stream input operator.

Instead, GiNaC lets you read an expression from a stream or a string, specifying the mapping between the input strings and symbols to be used:

```
{
    symbol x, y;
    symtab table;
    table["x"] = x;
    table["y"] = y;
    parser reader(table);
    ex e = reader("2*x+sin(y)");
}
```

The input syntax is the same as that used by `ginsh` and the stream output operator `<<`. Matching between the input strings and expressions is given by '`table`'. The '`table`' in this example instructs GiNaC to substitute any input substring "x" with symbol `x`. Likewise, the substring "y" will be replaced with symbol `y`. It's also possible to map input (sub)strings to arbitrary expressions:

```
{
    symbol x, y;
    symtab table;
    table["x"] = x+log(y)+1;
    parser reader(table);
    ex e = reader("5*x^3 - x^2");
    // e = 5*(x+log(y)+1)^3 - (x+log(y)+1)^2
}
```

If no mapping is specified for a particular string GiNaC will create a symbol with corresponding name. Later on you can obtain all parser generated symbols with `get_syms()` method:

```
{
    parser reader;
    ex e = reader("2*x+sin(y)");
    symtab table = reader.get_syms();
    symbol x = ex_to<symbol>(table["x"]);
    symbol y = ex_to<symbol>(table["y"]);
}
```

Sometimes you might want to prevent GiNaC from inserting these extra symbols (for example, you want treat an unexpected string in the input as an error).

```
{
symtab table;
table["x"] = symbol();
parser reader(table);
parser.strict = true;
ex e;
try {
e = reader("2*x+sin(y)");
} catch (parse_error& err) {
cerr << err.what() << endl;
// prints "unknown symbol "y" in the input"
}
}
```

With this parser, it's also easy to implement interactive GiNaC programs. When running the following program interactively, remember to send an EOF marker after the input, e.g. by pressing Ctrl-D on an empty line:

```
#include <iostream>
#include <string>
#include <stdexcept>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

int main()
{
cout << "Enter an expression containing 'x': " << flush;
parser reader;

try {
ex e = reader(cin);
symtab table = reader.get_syms();
symbol x = table.find("x") != table.end() ?
   ex_to<symbol>(table["x"]) : symbol("x");
cout << "The derivative of " << e << " with respect to x is ";
cout << e.diff(x) << "." << endl;
} catch (exception &p) {
cerr << p.what() << endl;
}
}
```

5.15.3 Compiling expressions to C function pointers

Numerical evaluation of algebraic expressions is seamlessly integrated into GiNaC by help of the CLN library. While CLN allows for very fast arbitrary precision numerics, which is more than sufficient for most users, sometimes only the speed of built-in floating point numbers is fast enough, e.g. for Monte Carlo integration. The only viable option then is the following: print the expression in C syntax format, manually add necessary C code, compile that program and run is as a separate application. This is not only cumbersome and involves a lot of manual intervention, but it also separates the algebraic and the numerical evaluation into different execution stages.

GiNaC offers a couple of functions that help to avoid these inconveniences and problems. The functions automatically perform the printing of a GiNaC expression and the subsequent compiling of its associated C code. The created object code is then dynamically linked to the currently running program. A function pointer to the C function that performs the numerical evaluation is returned and can be used instantly. This all happens automatically, no user intervention is needed.

The following example demonstrates the use of `compile_ex`:

```
// ...
symbol x("x");
ex myexpr = sin(x) / x;

FUNCP_1P fp;
compile_ex(myexpr, x, fp);

cout << fp(3.2) << endl;
// ...
```

The function `compile_ex` is called with the expression to be compiled and its only free variable `x`. Upon successful completion the third parameter contains a valid function pointer to the corresponding C code module. If called like in the last line only built-in double precision numerics is involved.

The function pointer has to be defined in advance. GiNaC offers three function pointer types at the moment:

```
typedef double (*FUNCP_1P) (double);
typedef double (*FUNCP_2P) (double, double);
typedef void (*FUNCP_CUBA) (const int*, const double[], const int*, double[]);
```

`FUNCP_2P` allows for two variables in the expression. `FUNCP_CUBA` is the correct type to be used with the CUBA library (`http://www.feynarts.de/cuba`) for numerical integrations. The details for the parameters of `FUNCP_CUBA` are explained in the CUBA manual.

For every function pointer type there is a matching `compile_ex` available:

```
void compile_ex(const ex& expr, const symbol& sym, FUNCP_1P& fp,
                const std::string filename = "");
void compile_ex(const ex& expr, const symbol& sym1, const symbol& sym2,
                FUNCP_2P& fp, const std::string filename = "");
void compile_ex(const lst& exprs, const lst& syms, FUNCP_CUBA& fp,
                const std::string filename = "");
```

When the last parameter `filename` is not supplied, `compile_ex` will choose a unique random name for the intermediate source and object files it produces. On program termination these files will be deleted. If one wishes to keep the C code and the object files, one can supply the `filename` parameter. The intermediate files will use that filename and will not be deleted.

`link_ex` is a function that allows to dynamically link an existing object file and to make it available via a function pointer. This is useful if you have already used `compile_ex` on an

expression and want to avoid the compilation step to be performed over and over again when you restart your program. The precondition for this is of course, that you have chosen a filename when you did call `compile_ex`. For every above mentioned function pointer type there exists a corresponding `link_ex` function:

```
void link_ex(const std::string filename, FUNCP_1P& fp);
void link_ex(const std::string filename, FUNCP_2P& fp);
void link_ex(const std::string filename, FUNCP_CUBA& fp);
```

The complete filename (including the suffix `.so`) of the object file has to be supplied.

The function

```
void unlink_ex(const std::string filename);
```

is supplied for the rare cases when one wishes to close the dynamically linked object files directly and have the intermediate files (only if filename has not been given) deleted. Normally one doesn't need this function, because all the clean-up will be done automatically upon (regular) program termination.

All the described functions will throw an exception in case they cannot perform correctly, like for example when writing the file or starting the compiler fails. Since internally the same printing methods as described in section [csrc printing], page 82, are used, only functions and objects that are available in standard C will compile successfully (that excludes polylogarithms for example at the moment). Another precondition for success is, of course, that it must be possible to evaluate the expression numerically. No free variables despite the ones supplied to `compile_ex` should appear in the expression.

`compile_ex` uses the shell script `ginac-excompiler` to start the C compiler and produce the object files. This shell script comes with GiNaC and will be installed together with GiNaC in the configured `$LIBEXECDIR` (typically `$PREFIX/libexec` or `$PREFIX/lib/ginac`). You can also export additional compiler flags via the `$CXXFLAGS` variable:

```
setenv("CXXFLAGS", "-O3 -fomit-frame-pointer", 1);
compile_ex(...);
```

5.15.4 Archiving

GiNaC allows creating *archives* of expressions which can be stored to or retrieved from files. To create an archive, you declare an object of class `archive` and archive expressions in it, giving each expression a unique name:

```
#include <fstream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

int main()
{
    symbol x("x"), y("y"), z("z");

    ex foo = sin(x + 2*y) + 3*z + 41;
    ex bar = foo + 1;

    archive a;
    a.archive_ex(foo, "foo");
    a.archive_ex(bar, "the second one");
    // ...
```

The archive can then be written to a file:

```
    // ...
```

```
    ofstream out("foobar.gar", ios::binary);
    out << a;
    out.close();
    // ...
```

The file `foobar.gar` contains all information that is needed to reconstruct the expressions `foo` and `bar`. The flag `ios::binary` prevents locales setting of your OS tampers the archive file structure.

The tool `viewgar` that comes with GiNaC can be used to view the contents of GiNaC archive files:

```
$ viewgar foobar.gar
foo = 41+sin(x+2*y)+3*z
the second one = 42+sin(x+2*y)+3*z
```

The point of writing archive files is of course that they can later be read in again:

```
    // ...
    archive a2;
    ifstream in("foobar.gar", ios::binary);
    in >> a2;
    // ...
```

And the stored expressions can be retrieved by their name:

```
    // ...
    lst syms = {x, y};

    ex ex1 = a2.unarchive_ex(syms, "foo");
    ex ex2 = a2.unarchive_ex(syms, "the second one");

    cout << ex1 << endl;              // prints "41+sin(x+2*y)+3*z"
    cout << ex2 << endl;              // prints "42+sin(x+2*y)+3*z"
    cout << ex1.subs(x == 2) << endl; // prints "41+sin(2+2*y)+3*z"
}
```

Note that you have to supply a list of the symbols which are to be inserted in the expressions. Symbols in archives are stored by their name only and if you don't specify which symbols you have, unarchiving the expression will create new symbols with that name. E.g. if you hadn't included `x` in the `syms` list above, the `ex1.subs(x == 2)` statement would have had no effect because the `x` in `ex1` would have been a different symbol than the `x` which was defined at the beginning of the program, although both would appear as '`x`' when printed.

You can also use the information stored in an `archive` object to output expressions in a format suitable for exact reconstruction. The `archive` and `archive_node` classes have a couple of member functions that let you access the stored properties:

```
static void my_print2(const archive_node & n)
{
    string class_name;
    n.find_string("class", class_name);
    cout << class_name << "(";

    archive_node::propinfovector p;
    n.get_properties(p);

    size_t num = p.size();
    for (size_t i=0; i<num; i++) {
        const string &name = p[i].name;
```

```
            if (name == "class")
                continue;
            cout << name << "=";

            unsigned count = p[i].count;
            if (count > 1)
                cout << "{";

            for (unsigned j=0; j<count; j++) {
                switch (p[i].type) {
                    case archive_node::PTYPE_BOOL: {
                        bool x;
                        n.find_bool(name, x, j);
                        cout << (x ? "true" : "false");
                        break;
                    }
                    case archive_node::PTYPE_UNSIGNED: {
                        unsigned x;
                        n.find_unsigned(name, x, j);
                        cout << x;
                        break;
                    }
                    case archive_node::PTYPE_STRING: {
                        string x;
                        n.find_string(name, x, j);
                        cout << '\"' << x << '\"';
                        break;
                    }
                    case archive_node::PTYPE_NODE: {
                        const archive_node &x = n.find_ex_node(name, j);
                        my_print2(x);
                        break;
                    }
                }

                if (j != count-1)
                    cout << ",";
            }

            if (count > 1)
                cout << "}";

            if (i != num-1)
                cout << ",";
        }

        cout << ")";
    }

    int main()
    {
        ex e = pow(2, x) - y;
```

```
    archive ar(e, "e");
    my_print2(ar.get_top_node(0)); cout << endl;
    return 0;
}
```

This will produce:

```
add(rest={power(basis=numeric(number="2"),exponent=symbol(name="x")),
symbol(name="y")},coeff={numeric(number="1"),numeric(number="-1")},
overall_coeff=numeric(number="0"))
```

Be warned, however, that the set of properties and their meaning for each class may change between GiNaC versions.

6 Extending GiNaC

By reading so far you should have gotten a fairly good understanding of GiNaC's design patterns. From here on you should start reading the sources. All we can do now is issue some recommendations how to tackle GiNaC's many loose ends in order to fulfill everybody's dreams. If you develop some useful extension please don't hesitate to contact the GiNaC authors—they will happily incorporate them into future versions.

6.1 What doesn't belong into GiNaC

First of all, GiNaC's name must be read literally. It is designed to be a library for use within C++. The tiny `ginsh` accompanying GiNaC makes this even more clear: it doesn't even attempt to provide a language. There are no loops or conditional expressions in `ginsh`, it is merely a window into the library for the programmer to test stuff (or to show off). Still, the design of a complete CAS with a language of its own, graphical capabilities and all this on top of GiNaC is possible and is without doubt a nice project for the future.

There are many built-in functions in GiNaC that do not know how to evaluate themselves numerically to a precision declared at runtime (using `Digits`). Some may be evaluated at certain points, but not generally. This ought to be fixed. However, doing numerical computations with GiNaC's quite abstract classes is doomed to be inefficient. For this purpose, the underlying foundation classes provided by CLN are much better suited.

6.2 Symbolic functions

The easiest and most instructive way to start extending GiNaC is probably to create your own symbolic functions. These are implemented with the help of two preprocessor macros:

```
DECLARE_FUNCTION_<n>P(<name>)
REGISTER_FUNCTION(<name>, <options>)
```

The `DECLARE_FUNCTION` macro will usually appear in a header file. It declares a C++ function with the given '`name`' that takes exactly '`n`' parameters of type `ex` and returns a newly constructed GiNaC `function` object that represents your function.

The `REGISTER_FUNCTION` macro implements the function. It must be passed the same '`name`' as the respective `DECLARE_FUNCTION` macro, and a set of options that associate the symbolic function with C++ functions you provide to implement the various methods such as evaluation, derivative, series expansion etc. They also describe additional attributes the function might have, such as symmetry and commutation properties, and a name for LaTeX output. Multiple options are separated by the member access operator '`.`' and can be given in an arbitrary order.

(By the way: in case you are worrying about all the macros above we can assure you that functions are GiNaC's most macro-intense classes. We have done our best to avoid macros where we can.)

6.2.1 A minimal example

Here is an example for the implementation of a function with two arguments that is not further evaluated:

```
DECLARE_FUNCTION_2P(myfcn)

REGISTER_FUNCTION(myfcn, dummy())
```

Any code that has seen the `DECLARE_FUNCTION` line can use `myfcn()` in algebraic expressions:

```
{
    ...
    symbol x("x");
```

```
    ex e = 2*myfcn(42, 1+3*x) - x;
    cout << e << endl;
     // prints '2*myfcn(42,1+3*x)-x'
    ...
}
```

The `dummy()` option in the `REGISTER_FUNCTION` line signifies "no options". A function with no options specified merely acts as a kind of container for its arguments. It is a pure "dummy" function with no associated logic (which is, however, sometimes perfectly sufficient).

Let's now have a look at the implementation of GiNaC's cosine function for an example of how to make an "intelligent" function.

6.2.2 The cosine function

The GiNaC header file `inifcns.h` contains the line

```
DECLARE_FUNCTION_1P(cos)
```

which declares to all programs using GiNaC that there is a function '`cos`' that takes one `ex` as an argument. This is all they need to know to use this function in expressions.

The implementation of the cosine function is in `inifcns_trans.cpp`. Here is its `REGISTER_FUNCTION` line:

```
REGISTER_FUNCTION(cos, eval_func(cos_eval).
                       evalf_func(cos_evalf).
                       derivative_func(cos_deriv).
                       latex_name("\\cos"));
```

There are four options defined for the cosine function. One of them (`latex_name`) gives the function a proper name for LaTeX output; the other three indicate the C++ functions in which the "brains" of the cosine function are defined.

The `eval_func()` option specifies the C++ function that implements the `eval()` method, GiNaC's anonymous evaluator. This function takes the same number of arguments as the associated symbolic function (one in this case) and returns the (possibly transformed or in some way simplified) symbolically evaluated function (See Section 4.2 [Automatic evaluation], page 10, for a description of the automatic evaluation process). If no (further) evaluation is to take place, the `eval_func()` function must return the original function with `.hold()`, to avoid a potential infinite recursion. If your symbolic functions produce a segmentation fault or stack overflow when using them in expressions, you are probably missing a `.hold()` somewhere.

The `eval_func()` function for the cosine looks something like this (actually, it doesn't look like this at all, but it should give you an idea what is going on):

```
static ex cos_eval(const ex & x)
{
    if ("x is a multiple of 2*Pi")
        return 1;
    else if ("x is a multiple of Pi")
        return -1;
    else if ("x is a multiple of Pi/2")
        return 0;
    // more rules...

    else if ("x has the form 'acos(y)'")
        return y;
    else if ("x has the form 'asin(y)'")
        return sqrt(1-y^2);
```

```
    // more rules...

    else
        return cos(x).hold();
}
```

This function is called every time the cosine is used in a symbolic expression:

```
{
    ...
    e = cos(Pi);
     // this calls cos_eval(Pi), and inserts its return value into
     // the actual expression
    cout << e << endl;
     // prints '-1'
    ...
}
```

In this way, `cos(4*Pi)` automatically becomes 1, `cos(asin(a+b))` becomes `sqrt(1-(a+b)^2)`, etc. If no reasonable symbolic transformation can be done, the unmodified function is returned with `.hold()`.

GiNaC doesn't automatically transform `cos(2)` to '`-0.416146...`'. The user has to call `evalf()` for that. This is implemented in a different function:

```
static ex cos_evalf(const ex & x)
{
    if (is_a<numeric>(x))
        return cos(ex_to<numeric>(x));
    else
        return cos(x).hold();
}
```

Since we are lazy we defer the problem of numeric evaluation to somebody else, in this case the `cos()` function for `numeric` objects, which in turn hands it over to the `cos()` function in CLN. The `.hold()` isn't really needed here, but reminds us that the corresponding `eval()` function would require it in this place.

Differentiation will surely turn up and so we need to tell `cos` what its first derivative is (higher derivatives, `.diff(x,3)` for instance, are then handled automatically by `basic::diff` and `ex::diff`):

```
static ex cos_deriv(const ex & x, unsigned diff_param)
{
    return -sin(x);
}
```

The second parameter is obligatory but uninteresting at this point. It specifies which parameter to differentiate in a partial derivative in case the function has more than one parameter, and its main application is for correct handling of the chain rule.

Derivatives of some functions, for example `abs()` and `Order()`, could not be evaluated through the chain rule. In such cases the full derivative may be specified as shown for `Order()`:

```
static ex Order_expl_derivative(const ex & arg, const symbol & s)
{
return Order(arg.diff(s));
}
```

That is, we need to supply a procedure, which returns the expression of derivative with respect to the variable `s` for the argument `arg`. This procedure need to be registered with the function

through the option `expl_derivative_func` (see the next Subsection). In contrast, a partial derivative, e.g. as was defined for `cos()` above, needs to be registered through the option `derivative_func`.

An implementation of the series expansion is not needed for `cos()` as it doesn't have any poles and GiNaC can do Taylor expansion by itself (as long as it knows what the derivative of `cos()` is). `tan()`, on the other hand, does have poles and may need to do Laurent expansion:

```
static ex tan_series(const ex & x, const relational & rel,
                     int order, unsigned options)
{
    // Find the actual expansion point
    const ex x_pt = x.subs(rel);

    if ("x_pt is not an odd multiple of Pi/2")
        throw do_taylor();  // tell function::series() to do Taylor expansion

    // On a pole, expand sin()/cos()
    return (sin(x)/cos(x)).series(rel, order+2, options);
}
```

The `series()` implementation of a function *must* return a `pseries` object, otherwise your code will crash.

6.2.3 Function options

GiNaC functions understand several more options which are always specified as `.option(params)`. None of them are required, but you need to specify at least one option to `REGISTER_FUNCTION()`. There is a do-nothing option called `dummy()` which you can use to define functions without any special options.

```
eval_func(<C++ function>)
evalf_func(<C++ function>)
derivative_func(<C++ function>)
expl_derivative_func(<C++ function>)
series_func(<C++ function>)
conjugate_func(<C++ function>)
```

These specify the C++ functions that implement symbolic evaluation, numeric evaluation, partial derivatives, explicit derivative, and series expansion, respectively. They correspond to the GiNaC methods `eval()`, `evalf()`, `diff()` and `series()`.

The `eval_func()` function needs to use `.hold()` if no further automatic evaluation is desired or possible.

If no `series_func()` is given, GiNaC defaults to simple Taylor expansion, which is correct if there are no poles involved. If the function has poles in the complex plane, the `series_func()` needs to check whether the expansion point is on a pole and fall back to Taylor expansion if it isn't. Otherwise, the pole usually needs to be regularized by some suitable transformation.

```
latex_name(const string & n)
```

specifies the LaTeX code that represents the name of the function in LaTeX output. The default is to put the function name in an `\mbox{}`.

```
do_not_evalf_params()
```

This tells `evalf()` to not recursively evaluate the parameters of the function before calling the `evalf_func()`.

```
set_return_type(unsigned return_type, const return_type_t * return_type_tinfo)
```

This allows you to explicitly specify the commutation properties of the function (See Section 4.15 [Non-commutative objects], page 39, for an explanation of (non)commutativity in GiNaC). For example, with an object of type `return_type_t` created like

```
return_type_t my_type = make_return_type_t<matrix>();
```

you can use `set_return_type(return_types::noncommutative, &my_type)` to make GiNaC treat your function like a matrix. By default, functions inherit the commutation properties of their first argument. The utilized template function `make_return_type_t<>()`

```
template<typename T> inline return_type_t make_return_type_t(const unsigned rl = 0)
```

can also be called with an argument specifying the representation label of the non-commutative function (see section on dirac gamma matrices for more details).

```
set_symmetry(const symmetry & s)
```

specifies the symmetry properties of the function with respect to its arguments. See Section 4.14 [Indexed objects], page 29, for an explanation of symmetry specifications. GiNaC will automatically rearrange the arguments of symmetric functions into a canonical order.

Sometimes you may want to have finer control over how functions are displayed in the output. For example, the `abs()` function prints itself as '`abs(x)`' in the default output format, but as '`|x|`' in LaTeX mode, and `fabs(x)` in C source output. This is achieved with the

```
print_func<C>(<C++ function>)
```

option which is explained in the next section.

6.2.4 Functions with a variable number of arguments

The `DECLARE_FUNCTION` and `REGISTER_FUNCTION` macros define functions with a fixed number of arguments. Sometimes, though, you may need to have a function that accepts a variable number of expressions. One way to accomplish this is to pass variable-length lists as arguments. The `Li()` function uses this method for multiple polylogarithms.

It is also possible to define functions that accept a different number of parameters under the same function name, such as the `psi()` function which can be called either as `psi(z)` (the digamma function) or as `psi(n, z)` (polygamma functions). These are actually two different functions in GiNaC that, however, have the same name. Defining such functions is not possible with the macros but requires manually fiddling with GiNaC internals. If you are interested, please consult the GiNaC source code for the `psi()` function (`inifcns.h` and `inifcns_gamma.cpp`).

6.3 GiNaC's expression output system

GiNaC allows the output of expressions in a variety of different formats (see Section 5.15 [Input/output], page 81). This section will explain how expression output is implemented internally, and how to define your own output formats or change the output format of built-in algebraic objects. You will also want to read this section if you plan to write your own algebraic classes or functions.

All the different output formats are represented by a hierarchy of classes rooted in the `print_context` class, defined in the `print.h` header file:

`print_dflt`
: the default output format

`print_latex`
: output in LaTeX mathematical mode

`print_tree`
: a dump of the internal expression structure (for debugging)

`print_csrc`
: the base class for C source output

`print_csrc_float`
: C source output using the `float` type

`print_csrc_double`
: C source output using the `double` type

`print_csrc_cl_N`
: C source output using CLN types

The `print_context` base class provides two public data members:

```
class print_context
{
    ...
public:
    std::ostream & s;
    unsigned options;
};
```

`s` is a reference to the stream to output to, while `options` holds flags and modifiers. Currently, there is only one flag defined: `print_options::print_index_dimensions` instructs the `idx` class to print the index dimension which is normally hidden.

When you write something like `std::cout << e`, where `e` is an object of class `ex`, GiNaC will construct an appropriate `print_context` object (of a class depending on the selected output format), fill in the `s` and `options` members, and call

```
void ex::print(const print_context & c, unsigned level = 0) const;
```

which in turn forwards the call to the `print()` method of the top-level algebraic object contained in the expression.

Unlike other methods, GiNaC classes don't usually override their `print()` method to implement expression output. Instead, the default implementation `basic::print(c, level)` performs a run-time double dispatch to a function selected by the dynamic type of the object and the passed `print_context`. To this end, GiNaC maintains a separate method table for each class, similar to the virtual function table used for ordinary (single) virtual function dispatch.

The method table contains one slot for each possible `print_context` type, indexed by the (internally assigned) serial number of the type. Slots may be empty, in which case GiNaC will retry the method lookup with the `print_context` object's parent class, possibly repeating the process until it reaches the `print_context` base class. If there's still no method defined, the method table of the algebraic object's parent class is consulted, and so on, until a matching method is found (eventually it will reach the combination `basic/print_context`, which prints the object's class name enclosed in square brackets).

You can think of the print methods of all the different classes and output formats as being arranged in a two-dimensional matrix with one axis listing the algebraic classes and the other axis listing the `print_context` classes.

Subclasses of `basic` can, of course, also overload `basic::print()` to implement printing, but then they won't get any of the benefits of the double dispatch mechanism (such as the ability for derived classes to inherit only certain print methods from its parent, or the replacement of methods at run-time).

6.3.1 Print methods for classes

The method table for a class is set up either in the definition of the class, by passing the appropriate `print_func<C>()` option to `GINAC_IMPLEMENT_REGISTERED_CLASS_OPT()` (See Section 6.5

[Adding classes], page 105, for an example), or at run-time using `set_print_func<T, C>()`. The latter can also be used to override existing methods dynamically.

The argument to `print_func<C>()` and `set_print_func<T, C>()` can be a member function of the class (or one of its parent classes), a static member function, or an ordinary (global) C++ function. The `C` template parameter specifies the appropriate `print_context` type for which the method should be invoked, while, in the case of `set_print_func<>()`, the `T` parameter specifies the algebraic class (for `print_func<>()`, the class is the one being implemented by `GINAC_IMPLEMENT_REGISTERED_CLASS_OPT`).

For print methods that are member functions, their first argument must be of a type convertible to a `const C &`, and the second argument must be an `unsigned`.

For static members and global functions, the first argument must be of a type convertible to a `const T &`, the second argument must be of a type convertible to a `const C &`, and the third argument must be an `unsigned`. A global function will, of course, not have access to private and protected members of `T`.

The `unsigned` argument of the print methods (and of `ex::print()` and `basic::print()`) is used for proper parenthesizing of the output (and by `print_tree` for proper indentation). It can be used for similar purposes if you write your own output formats.

The explanations given above may seem complicated, but in practice it's really simple, as shown in the following example. Suppose that we want to display exponents in LaTeX output not as superscripts but with little upwards-pointing arrows. This can be achieved in the following way:

```
void my_print_power_as_latex(const power & p,
                             const print_latex & c,
                             unsigned level)
{
    // get the precedence of the 'power' class
    unsigned power_prec = p.precedence();

    // if the parent operator has the same or a higher precedence
    // we need parentheses around the power
    if (level >= power_prec)
        c.s << '(';

    // print the basis and exponent, each enclosed in braces, and
    // separated by an uparrow
    c.s << '{';
    p.op(0).print(c, power_prec);
    c.s << "}\\uparrow{";
    p.op(1).print(c, power_prec);
    c.s << '}';

    // don't forget the closing parenthesis
    if (level >= power_prec)
        c.s << ')';
}

int main()
{
    // a sample expression
    symbol x("x"), y("y");
    ex e = -3*pow(x, 3)*pow(y, -2) + pow(x+y, 2) - 1;
```

```
    // switch to LaTeX mode
    cout << latex;

    // this prints "-1+{(y+x)}^{2}-3 \frac{x^{3}}{y^{2}}"
    cout << e << endl;

    // now we replace the method for the LaTeX output of powers with
    // our own one
    set_print_func<power, print_latex>(my_print_power_as_latex);

    // this prints "-1+{{(y+x)}}\uparrow{2}-3 \frac{{x}\uparrow{3}}{{y}
    //              \uparrow{2}}"
    cout << e << endl;
}
```

Some notes:

- The first argument of `my_print_power_as_latex` could also have been a `const basic &`, the second one a `const print_context &`.
- The above code depends on `mul` objects converting their operands to `power` objects for the purpose of printing.
- The output of products including negative powers as fractions is also controlled by the `mul` class.
- The `power/print_latex` method provided by GiNaC prints square roots using `\sqrt`, but the above code doesn't.

It's not possible to restore a method table entry to its previous or default value. Once you have called `set_print_func()`, you can only override it with another call to `set_print_func()`, but you can't easily go back to the default behavior again (you can, of course, dig around in the GiNaC sources, find the method that is installed at startup (`power::do_print_latex` in this case), and `set_print_func` that one; that is, after you circumvent the C++ member access control...).

6.3.2 Print methods for functions

Symbolic functions employ a print method dispatch mechanism similar to the one used for classes. The methods are specified with `print_func<C>()` function options. If you don't specify any special print methods, the function will be printed with its name (or LaTeX name, if supplied), followed by a comma-separated list of arguments enclosed in parentheses.

For example, this is what GiNaC's '`abs()`' function is defined like:

```
static ex abs_eval(const ex & arg) { ... }
static ex abs_evalf(const ex & arg) { ... }

static void abs_print_latex(const ex & arg, const print_context & c)
{
    c.s << "{|"; arg.print(c); c.s << "|}";
}

static void abs_print_csrc_float(const ex & arg, const print_context & c)
{
    c.s << "fabs("; arg.print(c); c.s << ")";
}
```

```
REGISTER_FUNCTION(abs, eval_func(abs_eval).
                       evalf_func(abs_evalf).
                       print_func<print_latex>(abs_print_latex).
                       print_func<print_csrc_float>(abs_print_csrc_float).
                       print_func<print_csrc_double>(abs_print_csrc_float));
```

This will display '`abs(x)`' as '`|x|`' in LaTeX mode and `fabs(x)` in non-CLN C source output, but as `abs(x)` in all other formats.

There is currently no equivalent of `set_print_func()` for functions.

6.3.3 Adding new output formats

Creating a new output format involves subclassing `print_context`, which is somewhat similar to adding a new algebraic class (see Section 6.5 [Adding classes], page 105). There is a macro `GINAC_DECLARE_PRINT_CONTEXT` that needs to go into the class definition, and a corresponding macro `GINAC_IMPLEMENT_PRINT_CONTEXT` that has to appear at global scope. Every `print_context` class needs to provide a default constructor and a constructor from an `std::ostream` and an `unsigned` options value.

Here is an example for a user-defined `print_context` class:

```
class print_myformat : public print_dflt
{
    GINAC_DECLARE_PRINT_CONTEXT(print_myformat, print_dflt)
public:
    print_myformat(std::ostream & os, unsigned opt = 0)
     : print_dflt(os, opt) {}
};

print_myformat::print_myformat() : print_dflt(std::cout) {}

GINAC_IMPLEMENT_PRINT_CONTEXT(print_myformat, print_dflt)
```

That's all there is to it. None of the actual expression output logic is implemented in this class. It merely serves as a selector for choosing a particular format. The algorithms for printing expressions in the new format are implemented as print methods, as described above.

`print_myformat` is a subclass of `print_dflt`, so it behaves exactly like GiNaC's default output format:

```
{
    symbol x("x");
    ex e = pow(x, 2) + 1;

    // this prints "1+x^2"
    cout << e << endl;

    // this also prints "1+x^2"
    e.print(print_myformat()); cout << endl;

    ...
}
```

To fill `print_myformat` with life, we need to supply appropriate print methods with `set_print_func()`, like this:

```
// This prints powers with '**' instead of '^'. See the LaTeX output
// example above for explanations.
void print_power_as_myformat(const power & p,
```

```
                                const print_myformat & c,
                                unsigned level)
{
    unsigned power_prec = p.precedence();
    if (level >= power_prec)
        c.s << '(';
    p.op(0).print(c, power_prec);
    c.s << "**";
    p.op(1).print(c, power_prec);
    if (level >= power_prec)
        c.s << ')';
}

{
    ...
    // install a new print method for power objects
    set_print_func<power, print_myformat>(print_power_as_myformat);

    // now this prints "1+x**2"
    e.print(print_myformat()); cout << endl;

    // but the default format is still "1+x^2"
    cout << e << endl;
}
```

6.4 Structures

If you are doing some very specialized things with GiNaC, or if you just need some more organized way to store data in your expressions instead of anonymous lists, you may want to implement your own algebraic classes. ('algebraic class' means any class directly or indirectly derived from `basic` that can be used in GiNaC expressions).

GiNaC offers two ways of accomplishing this: either by using the `structure<T>` template class, or by rolling your own class from scratch. This section will discuss the `structure<T>` template which is easier to use but more limited, while the implementation of custom GiNaC classes is the topic of the next section. However, you may want to read both sections because many common concepts and member functions are shared by both concepts, and it will also allow you to decide which approach is most suited to your needs.

The `structure<T>` template, defined in the GiNaC header file `structure.h`, wraps a type that you supply (usually a C++ `struct` or `class`) into a GiNaC object that can be used in expressions.

6.4.1 Example: scalar products

Let's suppose that we need a way to handle some kind of abstract scalar product of the form '`<x|y>`' in expressions. Objects of the scalar product class have to store their left and right operands, which can in turn be arbitrary expressions. Here is a possible way to represent such a product in a C++ `struct`:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

struct sprod_s {
```

```
    ex left, right;

    sprod_s() {}
    sprod_s(ex l, ex r) : left(l), right(r) {}
};
```

The default constructor is required. Now, to make a GiNaC class out of this data structure, we need only one line:

```
typedef structure<sprod_s> sprod;
```

That's it. This line constructs an algebraic class `sprod` which contains objects of type `sprod_s`. We can now use `sprod` in expressions like any other GiNaC class:

```
...
    symbol a("a"), b("b");
    ex e = sprod(sprod_s(a, b));
...
```

Note the difference between `sprod` which is the algebraic class, and `sprod_s` which is the unadorned C++ structure containing the `left` and `right` data members. As shown above, an `sprod` can be constructed from an `sprod_s` object.

If you find the nested `sprod(sprod_s())` constructor too unwieldy, you could define a little wrapper function like this:

```
inline ex make_sprod(ex left, ex right)
{
    return sprod(sprod_s(left, right));
}
```

The `sprod_s` object contained in `sprod` can be accessed with the GiNaC `ex_to<>()` function followed by the `->` operator or `get_struct()`:

```
...
    cout << ex_to<sprod>(e)->left << endl;
     // -> a
    cout << ex_to<sprod>(e).get_struct().right << endl;
     // -> b
...
```

You only have read access to the members of `sprod_s`.

The type definition of `sprod` is enough to write your own algorithms that deal with scalar products, for example:

```
ex swap_sprod(ex p)
{
    if (is_a<sprod>(p)) {
        const sprod_s & sp = ex_to<sprod>(p).get_struct();
        return make_sprod(sp.right, sp.left);
    } else
        return p;
}

...
    f = swap_sprod(e);
     // f is now <b|a>
...
```

6.4.2 Structure output

While the `sprod` type is useable it still leaves something to be desired, most notably proper output:

```
...
    cout << e << endl;
     // -> [structure object]
...
```

By default, any structure types you define will be printed as '`[structure object]`'. To override this you can either specialize the template's `print()` member function, or specify print methods with `set_print_func<>()`, as described in Section 6.3 [Printing], page 95. Unfortunately, it's not possible to supply class options like `print_func<>()` to structures, so for a self-contained structure type you need to resort to overriding the `print()` function, which is also what we will do here.

The member functions of GiNaC classes are described in more detail in the next section, but it shouldn't be hard to figure out what's going on here:

```
void sprod::print(const print_context & c, unsigned level) const
{
    // tree debug output handled by superclass
    if (is_a<print_tree>(c))
        inherited::print(c, level);

    // get the contained sprod_s object
    const sprod_s & sp = get_struct();

    // print_context::s is a reference to an ostream
    c.s << "<" << sp.left << "|" << sp.right << ">";
}
```

Now we can print expressions containing scalar products:

```
...
    cout << e << endl;
     // -> <a|b>
    cout << swap_sprod(e) << endl;
     // -> <b|a>
...
```

6.4.3 Comparing structures

The `sprod` class defined so far still has one important drawback: all scalar products are treated as being equal because GiNaC doesn't know how to compare objects of type `sprod_s`. This can lead to some confusing and undesired behavior:

```
...
    cout << make_sprod(a, b) - make_sprod(a*a, b*b) << endl;
     // -> 0
    cout << make_sprod(a, b) + make_sprod(a*a, b*b) << endl;
     // -> 2*<a|b> or 2*<a^2|b^2> (which one is undefined)
...
```

To remedy this, we first need to define the operators `==` and `<` for objects of type `sprod_s`:

```
inline bool operator==(const sprod_s & lhs, const sprod_s & rhs)
{
    return lhs.left.is_equal(rhs.left) && lhs.right.is_equal(rhs.right);
```

```
}

inline bool operator<(const sprod_s & lhs, const sprod_s & rhs)
{
    return lhs.left.compare(rhs.left) < 0
           ? true : lhs.right.compare(rhs.right) < 0;
}
```

The ordering established by the `<` operator doesn't have to make any algebraic sense, but it needs to be well defined. Note that we can't use expressions like `lhs.left == rhs.left` or `lhs.left < rhs.left` in the implementation of these operators because they would construct GiNaC `relational` objects which in the case of `<` do not establish a well defined ordering (for arbitrary expressions, GiNaC can't decide which one is algebraically 'less').

Next, we need to change our definition of the `sprod` type to let GiNaC know that an ordering relation exists for the embedded objects:

```
typedef structure<sprod_s, compare_std_less> sprod;
```

`sprod` objects then behave as expected:

```
...
    cout << make_sprod(a, b) - make_sprod(a*a, b*b) << endl;
     // -> <a|b>-<a^2|b^2>
    cout << make_sprod(a, b) + make_sprod(a*a, b*b) << endl;
     // -> <a|b>+<a^2|b^2>
    cout << make_sprod(a, b) - make_sprod(a, b) << endl;
     // -> 0
    cout << make_sprod(a, b) + make_sprod(a, b) << endl;
     // -> 2*<a|b>
...
```

The `compare_std_less` policy parameter tells GiNaC to use the `std::less` and `std::equal_to` functors to compare objects of type `sprod_s`. By default, these functors forward their work to the standard `<` and `==` operators, which we have overloaded. Alternatively, we could have specialized `std::less` and `std::equal_to` for class `sprod_s`.

GiNaC provides two other comparison policies for `structure<T>` objects: the default `compare_all_equal`, and `compare_bitwise` which does a bit-wise comparison of the contained `T` objects. This should be used with extreme care because it only works reliably with built-in integral types, and it also compares any padding (filler bytes of undefined value) that the `T` class might have.

6.4.4 Subexpressions

Our scalar product class has two subexpressions: the left and right operands. It might be a good idea to make them accessible via the standard `nops()` and `op()` methods:

```
size_t sprod::nops() const
{
    return 2;
}

ex sprod::op(size_t i) const
{
    switch (i) {
    case 0:
        return get_struct().left;
    case 1:
        return get_struct().right;
```

```
        default:
            throw std::range_error("sprod::op(): no such operand");
        }
    }
```

Implementing `nops()` and `op()` for container types such as `sprod` has two other nice side effects:

- `has()` works as expected
- GiNaC generates better hash keys for the objects (the default implementation of `calchash()` takes subexpressions into account)

There is a non-const variant of `op()` called `let_op()` that allows replacing subexpressions:

```
ex & sprod::let_op(size_t i)
{
    // every non-const member function must call this
    ensure_if_modifiable();

    switch (i) {
    case 0:
        return get_struct().left;
    case 1:
        return get_struct().right;
    default:
        throw std::range_error("sprod::let_op(): no such operand");
    }
}
```

Once we have provided `let_op()` we also get `subs()` and `map()` for free. In fact, every container class that returns a non-null `nops()` value must either implement `let_op()` or provide custom implementations of `subs()` and `map()`.

In turn, the availability of `map()` enables the recursive behavior of a couple of other default method implementations, in particular `evalf()`, `evalm()`, `normal()`, `diff()` and `expand()`. Although we probably want to provide our own version of `expand()` for scalar products that turns expressions like '`<a+b|c>`' into '`<a|c>+<b|c>`'. This is left as an exercise for the reader.

The `structure<T>` template defines many more member functions that you can override by specialization to customize the behavior of your structures. You are referred to the next section for a description of some of these (especially `eval()`). There is, however, one topic that shall be addressed here, as it demonstrates one peculiarity of the `structure<T>` template: archiving.

6.4.5 Archiving structures

If you don't know how the archiving of GiNaC objects is implemented, you should first read the next section and then come back here. You're back? Good.

To implement archiving for structures it is not enough to provide specializations for the `archive()` member function and the unarchiving constructor (the `unarchive()` function has a default implementation). You also need to provide a unique name (as a string literal) for each structure type you define. This is because in GiNaC archives, the class of an object is stored as a string, the class name.

By default, this class name (as returned by the `class_name()` member function) is '`structure`' for all structure classes. This works as long as you have only defined one structure type, but if you use two or more you need to provide a different name for each by specializing the `get_class_name()` member function. Here is a sample implementation for enabling archiving of the scalar product type defined above:

```
const char *sprod::get_class_name() { return "sprod"; }
```

```
void sprod::archive(archive_node & n) const
{
    inherited::archive(n);
    n.add_ex("left", get_struct().left);
    n.add_ex("right", get_struct().right);
}

sprod::structure(const archive_node & n, lst & sym_lst) : inherited(n, sym_lst)
{
    n.find_ex("left", get_struct().left, sym_lst);
    n.find_ex("right", get_struct().right, sym_lst);
}
```

Note that the unarchiving constructor is `sprod::structure` and not `sprod::sprod`, and that we don't need to supply an `sprod::unarchive()` function.

6.5 Adding classes

The `structure<T>` template provides an way to extend GiNaC with custom algebraic classes that is easy to use but has its limitations, the most severe of which being that you can't add any new member functions to structures. To be able to do this, you need to write a new class definition from scratch.

This section will explain how to implement new algebraic classes in GiNaC by giving the example of a simple 'string' class. After reading this section you will know how to properly declare a GiNaC class and what the minimum required member functions are that you have to implement. We only cover the implementation of a 'leaf' class here (i.e. one that doesn't contain subexpressions). Creating a container class like, for example, a class representing tensor products is more involved but this section should give you enough information so you can consult the source to GiNaC's predefined classes if you want to implement something more complicated.

6.5.1 Hierarchy of algebraic classes.

All algebraic classes (that is, all classes that can appear in expressions) in GiNaC are direct or indirect subclasses of the class `basic`. So a `basic *` represents a generic pointer to an algebraic class. Working with such pointers directly is cumbersome (think of memory management), hence GiNaC wraps them into `ex` (see Section A.1 [Expressions are reference counted], page 113). To make such wrapping possible every algebraic class has to implement several methods. Visitors (see Section 5.6 [Visitors and tree traversal], page 61), printing, and (un)archiving (see Section 5.15 [Input/output], page 81) require helper methods too. But don't worry, most of the work is simplified by the following macros (defined in `registrar.h`):

- `GINAC_DECLARE_REGISTERED_CLASS`
- `GINAC_IMPLEMENT_REGISTERED_CLASS`
- `GINAC_IMPLEMENT_REGISTERED_CLASS_OPT`

The `GINAC_DECLARE_REGISTERED_CLASS` macro inserts declarations required for memory management, visitors, printing, and (un)archiving. It takes the name of the class and its direct superclass as arguments. The `GINAC_DECLARE_REGISTERED_CLASS` should be the first line after the opening brace of the class definition.

`GINAC_IMPLEMENT_REGISTERED_CLASS` takes the same arguments as `GINAC_DECLARE_REGISTERED_CLASS`. It initializes certain static members of a class so that printing and (un)archiving works. The `GINAC_IMPLEMENT_REGISTERED_CLASS` may appear anywhere else in the source (at global scope, of course, not inside a function).

`GINAC_IMPLEMENT_REGISTERED_CLASS_OPT` is a variant of `GINAC_IMPLEMENT_REGISTERED_CLASS`. It allows specifying additional options, such as custom printing functions.

6.5.2 A minimalistic example

Now we will start implementing a new class `mystring` that allows placing character strings in algebraic expressions (this is not very useful, but it's just an example). This class will be a direct subclass of `basic`. You can use this sample implementation as a starting point for your own classes[1].

The code snippets given here assume that you have included some header files as follows:

```
#include <iostream>
#include <string>
#include <stdexcept>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;
```

Now we can write down the class declaration. The class stores a C++ `string` and the user shall be able to construct a `mystring` object from a string:

```
class mystring : public basic
{
    GINAC_DECLARE_REGISTERED_CLASS(mystring, basic)

public:
    mystring(const string & s);

private:
    string str;
};

GINAC_IMPLEMENT_REGISTERED_CLASS(mystring, basic)
```

The `GINAC_DECLARE_REGISTERED_CLASS` macro insert declarations required for memory management, visitors, printing, and (un)archiving. `GINAC_IMPLEMENT_REGISTERED_CLASS` initializes certain static members of a class so that printing and (un)archiving works.

Now there are three member functions we have to implement to get a working class:

- `mystring()`, the default constructor.
- `int compare_same_type(const basic & other)`, which is used internally by GiNaC to establish a canonical sort order for terms. It returns 0, +1 or -1, depending on the relative order of this object and the `other` object. If it returns 0, the objects are considered equal. **Please notice:** This has nothing to do with the (numeric) ordering relationship expressed by `<`, `>=` etc (which cannot be defined for non-numeric classes). For example, `numeric(1).compare_same_type(numeric(2))` may return +1 even though 1 is clearly smaller than 2. Every GiNaC class must provide a `compare_same_type()` function, even those representing objects for which no reasonable algebraic ordering relationship can be defined.
- And, of course, `mystring(const string& s)` which is the constructor we declared.

Let's proceed step-by-step. The default constructor looks like this:

```
mystring::mystring() { }
```

In the default constructor you should set all other member variables to reasonable default values (we don't need that here since our `str` member gets set to an empty string automatically).

[1] The self-contained source for this example is included in GiNaC, see the `doc/examples/mystring.cpp` file.

Our `compare_same_type()` function uses a provided function to compare the string members:

```
int mystring::compare_same_type(const basic & other) const
{
    const mystring &o = static_cast<const mystring &>(other);
    int cmpval = str.compare(o.str);
    if (cmpval == 0)
        return 0;
    else if (cmpval < 0)
        return -1;
    else
        return 1;
}
```

Although this function takes a `basic &`, it will always be a reference to an object of exactly the same class (objects of different classes are not comparable), so the cast is safe. If this function returns 0, the two objects are considered equal (in the sense that $A - B = 0$), so you should compare all relevant member variables.

Now the only thing missing is our constructor:

```
mystring::mystring(const string& s) : str(s) { }
```

No surprises here. We set the `str` member from the argument.

That's it! We now have a minimal working GiNaC class that can store strings in algebraic expressions. Let's confirm that the RTTI works:

```
ex e = mystring("Hello, world!");
cout << is_a<mystring>(e) << endl;
 // -> 1 (true)

cout << ex_to<basic>(e).class_name() << endl;
 // -> mystring
```

Obviously it does. Let's see what the expression `e` looks like:

```
cout << e << endl;
 // -> [mystring object]
```

Hm, not exactly what we expect, but of course the `mystring` class doesn't yet know how to print itself. This can be done either by implementing the `print()` member function, or, preferably, by specifying a `print_func<>()` class option. Let's say that we want to print the string surrounded by double quotes:

```
class mystring : public basic
{
    ...
protected:
    void do_print(const print_context & c, unsigned level = 0) const;
    ...
};

void mystring::do_print(const print_context & c, unsigned level) const
{
    // print_context::s is a reference to an ostream
    c.s << '\"' << str << '\"';
}
```

The `level` argument is only required for container classes to correctly parenthesize the output.

Now we need to tell GiNaC that `mystring` objects should use the `do_print()` member function for printing themselves. For this, we replace the line

```
GINAC_IMPLEMENT_REGISTERED_CLASS(mystring, basic)
```

with

```
GINAC_IMPLEMENT_REGISTERED_CLASS_OPT(mystring, basic,
  print_func<print_context>(&mystring::do_print))
```

Let's try again to print the expression:

```
cout << e << endl;
 // -> "Hello, world!"
```

Much better. If we wanted to have `mystring` objects displayed in a different way depending on the output format (default, LaTeX, etc.), we would have supplied multiple `print_func<>()` options with different template parameters (`print_dflt`, `print_latex`, etc.), separated by dots. This is similar to the way options are specified for symbolic functions. See Section 6.3 [Printing], page 95, for a more in-depth description of the way expression output is implemented in GiNaC.

The `mystring` class can be used in arbitrary expressions:

```
e += mystring("GiNaC rulez");
cout << e << endl;
 // -> "GiNaC rulez"+"Hello, world!"
```

(GiNaC's automatic term reordering is in effect here), or even

```
e = pow(mystring("One string"), 2*sin(Pi-mystring("Another string")));
cout << e << endl;
 // -> "One string"^(2*sin(-"Another string"+Pi))
```

Whether this makes sense is debatable but remember that this is only an example. At least it allows you to implement your own symbolic algorithms for your objects.

Note that GiNaC's algebraic rules remain unchanged:

```
e = mystring("Wow") * mystring("Wow");
cout << e << endl;
 // -> "Wow"^2

e = pow(mystring("First")-mystring("Second"), 2);
cout << e.expand() << endl;
 // -> -2*"First"*"Second"+"First"^2+"Second"^2
```

There's no way to, for example, make GiNaC's `add` class perform string concatenation. You would have to implement this yourself.

6.5.3 Automatic evaluation

When dealing with objects that are just a little more complicated than the simple string objects we have implemented, chances are that you will want to have some automatic simplifications or canonicalizations performed on them. This is done in the evaluation member function `eval()`. Let's say that we wanted all strings automatically converted to lowercase with non-alphabetic characters stripped, and empty strings removed:

```
class mystring : public basic
{
    ...
public:
    ex eval() const override;
    ...
};
```

```
ex mystring::eval() const
{
    string new_str;
    for (size_t i=0; i<str.length(); i++) {
        char c = str[i];
        if (c >= 'A' && c <= 'Z')
            new_str += tolower(c);
        else if (c >= 'a' && c <= 'z')
            new_str += c;
    }

    if (new_str.length() == 0)
        return 0;

    return mystring(new_str).hold();
}
```

The `hold()` member function sets a flag in the object that prevents further evaluation. Otherwise we might end up in an endless loop. When you want to return the object unmodified, use `return this->hold();`.

If our class had subobjects, we would have to evaluate them first (unless they are all of type `ex`, which are automatically evaluated). We don't have any subexpressions in the `mystring` class, so we are not concerned with this.

Let's confirm that it works:

```
ex e = mystring("Hello, world!") + mystring("!?#");
cout << e << endl;
 // -> "helloworld"

e = mystring("Wow!") + mystring("WOW") + mystring(" W ** o ** W");
cout << e << endl;
 // -> 3*"wow"
```

6.5.4 Optional member functions

We have implemented only a small set of member functions to make the class work in the GiNaC framework. There are two functions that are not strictly required but will make operations with objects of the class more efficient:

```
unsigned calchash() const override;
bool is_equal_same_type(const basic & other) const override;
```

The `calchash()` method returns an `unsigned` hash value for the object which will allow GiNaC to compare and canonicalize expressions much more efficiently. You should consult the implementation of some of the built-in GiNaC classes for examples of hash functions. The default implementation of `calchash()` calculates a hash value out of the `tinfo_key` of the class and all subexpressions that are accessible via `op()`.

`is_equal_same_type()` works like `compare_same_type()` but only tests for equality without establishing an ordering relation, which is often faster. The default implementation of `is_equal_same_type()` just calls `compare_same_type()` and tests its result for zero.

6.5.5 Other member functions

For a real algebraic class, there are probably some more functions that you might want to provide:

```
bool info(unsigned inf) const override;
ex evalf() const override;
ex series(const relational & r, int order, unsigned options = 0) const override;
ex derivative(const symbol & s) const override;
```

If your class stores sub-expressions (see the scalar product example in the previous section) you will probably want to override

```
size_t nops() const override;
ex op(size_t i) const override;
ex & let_op(size_t i) override;
ex subs(const lst & ls, const lst & lr, unsigned options = 0) const override;
ex map(map_function & f) const override;
```

`let_op()` is a variant of `op()` that allows write access. The default implementations of `subs()` and `map()` use it, so you have to implement either `let_op()`, or `subs()` and `map()`.

You can, of course, also add your own new member functions. Remember that the RTTI may be used to get information about what kinds of objects you are dealing with (the position in the class hierarchy) and that you can always extract the bare object from an `ex` by stripping the `ex` off using the `ex_to<mystring>(e)` function when that should become a need.

That's it. May the source be with you!

6.5.6 Upgrading extension classes from older version of GiNaC

GiNaC used to use a custom run time type information system (RTTI). It was removed from GiNaC. Thus, one needs to rewrite constructors which set `tinfo_key` (which does not exist any more). For example,

```
myclass::myclass() : inherited(&myclass::tinfo_static) {}
```

needs to be rewritten as

```
myclass::myclass() {}
```

7 A Comparison With Other CAS

This chapter will give you some information on how GiNaC compares to other, traditional Computer Algebra Systems, like *Maple*, *Mathematica* or *Reduce*, where it has advantages and disadvantages over these systems.

7.1 Advantages

GiNaC has several advantages over traditional Computer Algebra Systems, like

- familiar language: all common CAS implement their own proprietary grammar which you have to learn first (and maybe learn again when your vendor decides to 'enhance' it). With GiNaC you can write your program in common C++, which is standardized.
- structured data types: you can build up structured data types using `struct`s or `class`es together with STL features instead of using unnamed lists of lists of lists.
- strongly typed: in CAS, you usually have only one kind of variables which can hold contents of an arbitrary type. This 4GL like feature is nice for novice programmers, but dangerous.
- development tools: powerful development tools exist for C++, like fancy editors (e.g. with automatic indentation and syntax highlighting), debuggers, visualization tools, documentation generators...
- modularization: C++ programs can easily be split into modules by separating interface and implementation.
- price: GiNaC is distributed under the GNU Public License which means that it is free and available with source code. And there are excellent C++-compilers for free, too.
- extendable: you can add your own classes to GiNaC, thus extending it on a very low level. Compare this to a traditional CAS that you can usually only extend on a high level by writing in the language defined by the parser. In particular, it turns out to be almost impossible to fix bugs in a traditional system.
- multiple interfaces: Though real GiNaC programs have to be written in some editor, then be compiled, linked and executed, there are more ways to work with the GiNaC engine. Many people want to play with expressions interactively, as in traditional CASs: The tiny `ginsh` that comes with the distribution exposes many, but not all, of GiNaC's types to a command line.
- seamless integration: it is somewhere between difficult and impossible to call CAS functions from within a program written in C++ or any other programming language and vice versa. With GiNaC, your symbolic routines are part of your program. You can easily call third party libraries, e.g. for numerical evaluation or graphical interaction. All other approaches are much more cumbersome: they range from simply ignoring the problem (i.e. *Maple*) to providing a method for 'embedding' the system (i.e. *Yacas*).
- efficiency: often large parts of a program do not need symbolic calculations at all. Why use large integers for loop variables or arbitrary precision arithmetics where `int` and `double` are sufficient? For pure symbolic applications, GiNaC is comparable in speed with other CAS.

7.2 Disadvantages

GiNaC cannot compete with a program like *Reduce* or *Maple* which exists for more than 50 years now with respect to some mathematical features. Integration, non-trivial simplifications, limits etc. are missing in GiNaC (and are not planned for the near future).

7.3 Why C++?

Why did we choose to implement GiNaC in C++ instead of Java or any other language? C++ is not perfect: type checking is not strict (casting is possible), separation between interface and implementation is not complete, object oriented design is not enforced. The main reason is the often scolded feature of operator overloading in C++. While it may be true that operating on classes with a `+` operator is rarely meaningful, it is perfectly suited for algebraic expressions. Writing $3x+5y$ as `3*x+5*y` instead of `x.times(3).plus(y.times(5))` looks much more natural. Furthermore, the main developers are more familiar with C++ than with any other programming language.

Appendix A Internal structures

A.1 Expressions are reference counted

In GiNaC, there is an *intrusive reference-counting* mechanism at work where the counter belongs to the algebraic objects derived from class `basic` but is maintained by the smart pointer class `ptr`, of which `ex` contains an instance. If you understood that, you can safely skip the rest of this passage.

Expressions are extremely light-weight since internally they work like handles to the actual representation. They really hold nothing more than a pointer to some other object. What this means in practice is that whenever you create two `ex` and set the second equal to the first no copying process is involved. Instead, the copying takes place as soon as you try to change the second. Consider the simple sequence of code:

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

int main()
{
    symbol x("x"), y("y"), z("z");
    ex e1, e2;

    e1 = sin(x + 2*y) + 3*z + 41;
    e2 = e1;                // e2 points to same object as e1
    cout << e2 << endl;     // prints sin(x+2*y)+3*z+41
    e2 += 1;                // e2 is copied into a new object
    cout << e2 << endl;     // prints sin(x+2*y)+3*z+42
}
```

The line `e2 = e1;` creates a second expression pointing to the object held already by `e1`. The time involved for this operation is therefore constant, no matter how large `e1` was. Actual copying, however, must take place in the line `e2 += 1;` because `e1` and `e2` are not handles for the same object any more. This concept is called *copy-on-write semantics*. It increases performance considerably whenever one object occurs multiple times and represents a simple garbage collection scheme because when an `ex` runs out of scope its destructor checks whether other expressions handle the object it points to too and deletes the object from memory if that turns out not to be the case. A slightly less trivial example of differentiation using the chain-rule should make clear how powerful this can be:

```
{
    symbol x("x"), y("y");

    ex e1 = x + 3*y;
    ex e2 = pow(e1, 3);
    ex e3 = diff(sin(e2), x);   // first derivative of sin(e2) by x
    cout << e1 << endl          // prints x+3*y
         << e2 << endl          // prints (x+3*y)^3
         << e3 << endl;         // prints 3*(x+3*y)^2*cos((x+3*y)^3)
}
```

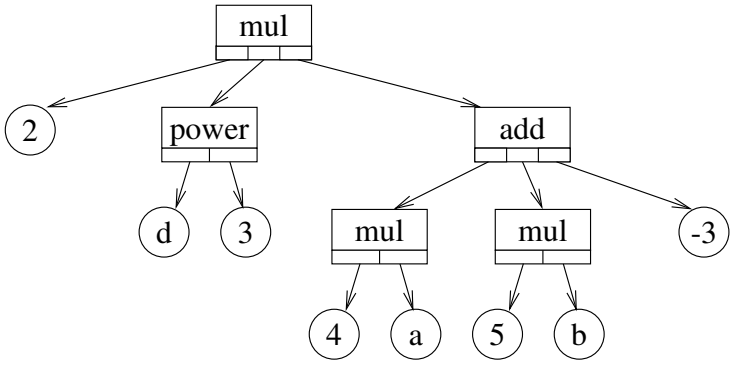
Here, `e1` will actually be referenced three times while `e2` will be referenced two times. When the power of an expression is built, that expression needs not be copied. Likewise, since the

derivative of a power of an expression can be easily expressed in terms of that expression, no copying of `e1` is involved when `e3` is constructed. So, when `e3` is constructed it will print as `3*(x+3*y)^2*cos((x+3*y)^3)` but the argument of `cos()` only holds a reference to `e2` and the factor in front is just `3*e1^2`.

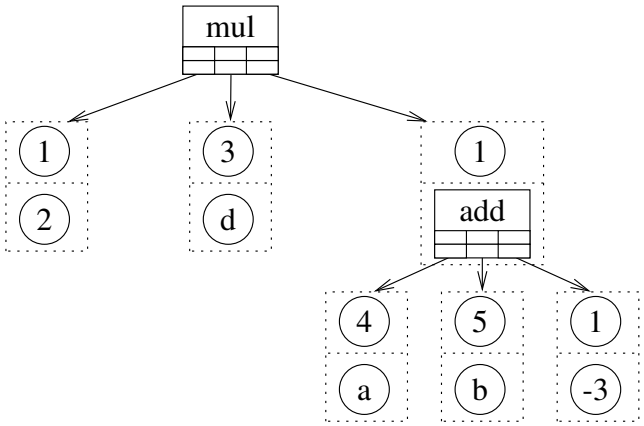
As a user of GiNaC, you cannot see this mechanism of copy-on-write semantics. When you insert an expression into a second expression, the result behaves exactly as if the contents of the first expression were inserted. But it may be useful to remember that this is not what happens. Knowing this will enable you to write much more efficient code. If you still have an uncertain feeling with copy-on-write semantics, we recommend you have a look at the C++-FAQ's (`https://isocpp.org/faq`) chapter on memory management. It covers this issue and presents an implementation which is pretty close to the one in GiNaC.

A.2 Internal representation of products and sums

Although it should be completely transparent for the user of GiNaC a short discussion of this topic helps to understand the sources and also explain performance to a large degree. Consider the unexpanded symbolic expression $2d^3\,(4a+5b-3)$ which could naively be represented by a tree of linear containers for addition and multiplication, one container for exponentiation with base and exponent and some atomic leaves of symbols and numbers in this fashion:

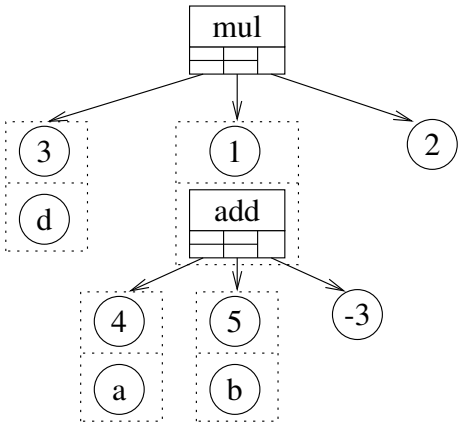


However, doing so results in a rather deeply nested tree which will quickly become inefficient to manipulate. We can improve on this by representing the sum as a sequence of terms, each one being a pair of a purely numeric multiplicative coefficient and its rest. In the same spirit we can store the multiplication as a sequence of terms, each having a numeric exponent and a possibly complicated base, the tree becomes much more flat:



The number `3` above the symbol `d` shows that `mul` objects are treated similarly where the coefficients are interpreted as *exponents* now. Addition of sums of terms or multiplication of products with numerical exponents can be coded to be very efficient with such a pair-wise

representation. Internally, this handling is performed by most CAS in this way. It typically speeds up manipulations by an order of magnitude. The overall multiplicative factor `2` and the additive term `-3` look somewhat out of place in this representation, however, since they are still carrying a trivial exponent and multiplicative factor `1` respectively. Within GiNaC, this is avoided by adding a field that carries an overall numeric coefficient. This results in the realistic picture of internal representation for $2d^3\,(4a+5b-3)$:



This also allows for a better handling of numeric radicals, since `sqrt(2)` can now be carried along calculations. Now it should be clear, why both classes `add` and `mul` are derived from the same abstract class: the data representation is the same, only the semantics differs. In the class hierarchy, methods for polynomial expansion and the like are reimplemented for `add` and `mul`, but the data structure is inherited from `expairseq`.

Appendix B Package tools

If you are creating a software package that uses the GiNaC library, setting the correct command line options for the compiler and linker can be difficult. The `pkgconf` utility makes this process easier. GiNaC supplies all necessary data in `ginac.pc` (installed into *LIBDIR*`/pkgconfig` by default). To compile a simple program use[1]

```
g++ -o simple simple.cpp `pkgconf --cflags --libs ginac`
```

This command line might expand to (for example):

```
g++ -o simple simple.cpp -lginac -lcln
```

Not only is the form using `pkgconf` easier to type, it will work on any system, no matter how GiNaC was configured.

For packages configured using GNU automake, `pkgconf` also provides the `PKG_CHECK_MODULES` macro to automate the process of checking for libraries

```
PKG_CHECK_MODULES(MYAPP, ginac >= MINIMUM_VERSION,
                  [ACTION-IF-FOUND],
                  [ACTION-IF-NOT-FOUND])
```

This macro:

- Determines the location of GiNaC using data from `ginac.pc`, which is either found in the default `pkgconf` search path, or from the environment variable `PKG_CONFIG_PATH`.
- Tests the installed libraries to make sure that their version is later than *MINIMUM-VERSION*.
- If the required version was found, sets the `MYAPP_CFLAGS` variable to the output of `pkgconf --cflags ginac` and the `MYAPP_LIBS` variable to the output of `pkgconf --libs ginac`, and calls '`AC_SUBST()`' for these variables so they can be used in generated makefiles, and then executes *ACTION-IF-FOUND*.
- If the required version was not found, executes *ACTION-IF-NOT-FOUND*.

B.1 Configuring a package that uses GiNaC

The directory where the GiNaC libraries are installed needs to be found by your system's dynamic linkers (both compile- and run-time ones). See the documentation of your system linker for details. Also make sure that `ginac.pc` is in `pkgconf`'s search path, See Section "pkgconf" in `*manpages*`.

The short summary below describes how to do this on a GNU/Linux system.

Suppose GiNaC is installed into the directory *PREFIX*. To tell the linkers where to find the library one should

- edit `/etc/ld.so.conf` and run `ldconfig`. For example,

  ```
  # echo PREFIX/lib >> /etc/ld.so.conf
  # ldconfig
  ```

- or set the environment variables `LD_LIBRARY_PATH` and `LD_RUN_PATH`

  ```
  $ export LD_LIBRARY_PATH=PREFIX/lib
  $ export LD_RUN_PATH=PREFIX/lib
  ```

- or give a '`-L`' and '`--rpath`' flags when running configure, for instance:

  ```
  $ LDFLAGS='-Wl,-LPREFIX/lib -Wl,--rpath=PREFIX/lib' ./configure
  ```

[1] If GiNaC is installed into some non-standard directory *prefix* one should set the *PKG_CONFIG_PATH* environment variable to *LIBDIR*`/pkgconfig` for this to work.

To tell `pkgconf` where the `ginac.pc` file is, set the `PKG_CONFIG_PATH` environment variable:

```
$ export PKG_CONFIG_PATH=PREFIX/lib/pkgconfig
```

Finally, run the `configure` script

```
$ ./configure
```

B.2 Example of a package using GiNaC

The following shows how to build a simple package using automake and the '`PKG_CHECK_MODULES`' macro. The program used here is `simple.cpp`:

```
#include <iostream>
#include <ginac/ginac.h>

int main()
{
    GiNaC::symbol x("x");
    GiNaC::ex a = GiNaC::sin(x);
    std::cout << "Derivative of " << a
              << " is " << a.diff(x) << std::endl;
    return 0;
}
```

You should first read the introductory portions of the automake manual, if you are not already familiar with it.

Two files are needed, `configure.ac`, which is used to build the configure script:

```
dnl Process this file with autoreconf to produce a configure script.
AC_INIT([simple], [1.0.0], [simple@example.net])
AC_CONFIG_SRCDIR(simple.cpp)
AM_INIT_AUTOMAKE

AC_PROG_CXX
AC_PROG_INSTALL
AC_LANG([C++])

PKG_CHECK_MODULES(SIMPLE, ginac >= 1.8.0)

AC_CONFIG_FILES([Makefile])
AC_OUTPUT
```

The '`PKG_CHECK_MODULES`' macro does the following: If a GiNaC version greater or equal than 1.8.0 is found, then it defines *SIMPLE_CFLAGS* and *SIMPLE_LIBS*. Otherwise, it dies with the error message like

```
configure: error: Package requirements (ginac >= 1.8.0) were not met:

Package dependency requirement 'ginac >= 1.8.0' could not be satisfied.
Package 'ginac' has version '1.7.7', required version is '>= 1.8.0'

Consider adjusting the PKG_CONFIG_PATH environment variable if you
installed software in a non-standard prefix.

Alternatively, you may set the environment variables SIMPLE_CFLAGS
and SIMPLE_LIBS to avoid the need to call pkg-config.
See the pkg-config man page for more details.
```

And the `Makefile.am`, which will be used to build the Makefile.

```
## Process this file with automake to produce Makefile.in
bin_PROGRAMS = simple
simple_SOURCES = simple.cpp
simple_CPPFLAGS = $(SIMPLE_CFLAGS)
simple_LDADD = $(SIMPLE_LIBS)
```

This `Makefile.am`, says that we are building a single executable, from a single source file `simple.cpp`. Since every program we are building uses GiNaC we could have simply added *SIMPLE_CFLAGS* to *CPPFLAGS* and *SIMPLE_LIBS* to *LIBS*. However, it is more flexible to specify libraries and complier options on a per-program basis.

To try this example out, create a new directory and add the three files above to it.

Now execute the following command:

```
$ autoreconf -i
```

You now have a package that can be built in the normal fashion

```
$ ./configure
$ make
$ make install
```

Appendix C Bibliography

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